

# SEMI-ORTHOGONAL DECOMPOSITION OF GIT QUOTIENT STACKS

ŠPELA ŠPENKO AND MICHEL VAN DEN BERGH

**ABSTRACT.** If  $G$  is a reductive group which acts on a linearized smooth scheme  $X$  then we show that under suitable standard conditions the derived category  $\mathcal{D}(X^{ss}/G)$  of the corresponding GIT quotient stack  $X^{ss}/G$  has a semi-orthogonal decomposition consisting of derived categories of coherent sheaves of rings on  $X^{ss}/G$  which are locally of finite global dimension. One of the components of the decomposition is a certain non-commutative resolution of  $X^{ss}/G$  constructed earlier by the authors.

The results in this paper also complement a result by Halpern-Leistner (and similar results by Ballard-Favero-Katzarkov and Donovan-Segal) that asserts the existence of a semi-orthogonal decomposition of  $\mathcal{D}(X/G)$  in which one of the parts is  $\mathcal{D}(X^{ss}/G)$ .

## 1. INTRODUCTION

**1.1. Main result.** Throughout  $k$  is an algebraically closed base field of characteristic 0. All schemes are  $k$ -schemes. If  $\Lambda$  is a right noetherian ring then we write  $\mathcal{D}(\Lambda)$  for  $D_f^b(\Lambda) \subset D(\Lambda)$ , the bounded derived category of right  $\Lambda$ -modules with finitely generated cohomology. Similarly for a noetherian scheme/stack  $\mathcal{X}$  we write  $\mathcal{D}(\mathcal{X}) := D_{\text{coh}}^b(\mathcal{X})$  and if  $\mathcal{A}$  is a quasi-coherent sheaf of noetherian algebras on a stack  $\mathcal{X}$  then we write  $\mathcal{D}(\mathcal{A})$  for  $D_{\text{coh}}^b(\mathcal{A})$ .

**Definition 1.1.1.** Let  $\mathcal{D}$  be a triangulated category. A *semi-orthogonal decomposition*  $\mathcal{D} = \langle \mathcal{D}_i \mid i \in I \rangle$  is a list of triangulated subcategories  $(\mathcal{D}_i)_{i \in I}$  of  $\mathcal{D}$  indexed by a totally ordered set  $I$  such that

- (1)  $\mathcal{D}$  is the smallest thick subcategory of  $\mathcal{D}$  containing  $\mathcal{D}_i$ ,  $i \in I$ ,
- (2)  $\text{Hom}(\mathcal{D}_i, \mathcal{D}_j) = 0$  for  $j < i$ .

Let  $X$  be a scheme. A *presheaf of triangulated categories*  $\tilde{\mathcal{E}}$  on  $X$  consists of triangulated categories  $\tilde{\mathcal{E}}(U)$  for all open subschemes  $U \subset X$  together with exact restriction functors  $\tilde{\mathcal{E}}(\bar{U}) \rightarrow \tilde{\mathcal{E}}(\bar{V})$  for  $V \subset U$  satisfying the usual compatibilities. A triangulated subpresheaf  $\tilde{\mathcal{F}}$  of  $\tilde{\mathcal{E}}$  is a collection of triangulated subcategories  $\tilde{\mathcal{F}}(U) \subset \tilde{\mathcal{E}}(U)$  compatible with restriction.

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A semi-orthogonal decomposition  $\tilde{\mathcal{E}} = \langle \tilde{\mathcal{E}}_i \mid i \in I \rangle$  is a list of triangulated sub-presheaves  $(\tilde{\mathcal{E}}_i)_{i \in I}$  of  $\tilde{\mathcal{E}}$  indexed by a totally ordered set  $I$  such that for each open  $U \subset X$  we have a semi-orthogonal decomposition  $\tilde{\mathcal{E}}(U) = \langle \tilde{\mathcal{E}}_i(U) \mid i \in I \rangle$ .

If  $X$  is noetherian then we write  $\tilde{\mathcal{D}}_X$  for the presheaf of triangulated categories  $U \mapsto \mathcal{D}(U)$  on  $X$ . For a quasi-coherent sheaf of noetherian algebras  $\mathcal{A}$  on  $X$  we similarly put  $\tilde{\mathcal{D}}_{\mathcal{A}}(U) = \mathcal{D}(\mathcal{A} \mid U)$ .

Let  $G$  be a reductive group acting on a  $k$ -scheme  $X$  such that a good quotient  $\pi : X \rightarrow X//G$  exists (see §3.3 below). Then we define a presheaf of triangulated categories  $\tilde{\mathcal{D}}_{X/G}$  on  $X//G$  as follows: if  $U \subset X//G$  is open then we put  $\tilde{\mathcal{D}}_{X/G}(U) = \mathcal{D}((U \times_{X//G} X)/G)$ .

**Theorem 1.1.2.** *Let  $G$  be a reductive group acting on a smooth variety  $X$  such that a good quotient  $\pi : X \rightarrow X//G$  exists and put  $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_{X/G}$ . There exist one-parameter subgroups  $\lambda_i : G_m \rightarrow G$  of  $G$ , open subgroups  $\tilde{G}^{\lambda_i}$  of  $G^{\lambda_i}$  and finite dimensional  $\tilde{G}^{\lambda_i}$ -representations  $U_i$ , such that  $\tilde{\mathcal{D}} = \langle \dots, \tilde{\mathcal{D}}_{-2}, \tilde{\mathcal{D}}_{-1}, \tilde{\mathcal{D}}_0 \rangle$  with  $\tilde{\mathcal{D}}_{-i} \cong \tilde{\mathcal{D}}_{\Lambda_i}$  for sheaves of  $\mathcal{O}_{X^{\lambda_i} // \tilde{G}^{\lambda_i}}$ -algebras (viewed as  $\mathcal{O}_{X//G}$ -algebras) defined by  $\Lambda_i \cong (\text{End}(U_i) \otimes_k \pi_* \mathcal{O}_{X^{\lambda_i}})^{\tilde{G}^{\lambda_i}}$ . The restrictions of  $\Lambda_i$  to affine opens have finite global dimension.*

In this theorem the notation  $(-)^{\lambda_i}$  was used for the fixed points under  $\lambda_i$  (see §3.3 below). Note that  $G^{\lambda_i}$  is a reductive subgroup of  $G$  (see §3.6) acting on  $X^{\lambda_i}$ .

Theorem 1.1.2 applies in particular to GIT stack quotients of the form  $X^{ss}/G$  where  $X$  is a smooth projective variety over an affine variety equipped with an ample linearization. In that way Theorem 1.1.2 complements [HL15, Theorem 2.10] (and similar results in [BFK12, DS14]) which constructs a semi-orthogonal decomposition of  $\mathcal{D}(X/G)$  in which one of the parts is  $\mathcal{D}(X^{ss}/G)$ .

**1.2. The linear case.** The proof of Theorem 1.1.2 will be reduced ultimately to the case that  $G$  is connected and  $X$  is a representation. In this section we give a more precise description of the semi-orthogonal decomposition in this case.

We first need to introduce more notation. Let  $T \subset B$  be a maximal torus and a Borel subgroup in  $G$ . Let  $X(T)$  and  $Y(T) = X(T)^\vee$  be respectively the character group of  $T$  and the group of one-parameter subgroups of  $T$ . Let the roots of  $B$  be the negative roots and let  $X(T)^\pm, Y(T)^\pm$  be the (anti-)dominant cones in  $X(T)$  and  $Y(T)$ . Let  $\bar{\rho} \in X(T)_\mathbb{R}$  be half the sum of the positive roots.

Let  $W$  be a finite dimensional  $G$ -representation of dimension  $d$  such that  $X = W^\vee$  and let  $R = k[X] = \text{Sym}(W)$ . Let  $(\beta_i)_{i=1}^d \in X(T)$  be the  $T$ -weights of  $W$ . For  $\lambda \in Y(T)^\vee$  define the following subsets of  $X(T)_\mathbb{R}$

$$\Sigma_\lambda = \left\{ \sum_i a_i \beta_i \mid a_i \in [-1, 0], \langle \lambda, \beta_i \rangle = 0 \right\}, \quad \Sigma := \Sigma_0,$$

$$\Gamma = -\bar{\rho} + \left\{ \sum_i a_i \beta_i \mid a_i \leq 0 \right\}.$$

We denote  $\Sigma_\lambda^0 = \text{relint} \Sigma_\lambda = \{ \sum_i a_i \beta_i \mid a_i \in ]-1, 0[, \langle \lambda, \beta_i \rangle = 0 \}$ ,  $\Gamma^0 = \text{relint} \Gamma$ . With  $W_\lambda$  we denote quotient space of  $W$  obtained by dividing out the weight vectors  $w_i$

such that  $\langle \lambda, \beta_i \rangle \neq 0$ . We further denote (see also §3.6 below):

$$G^{\lambda,+} = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \},$$

$$G^\lambda = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = g\}.$$

(Note that  $G^\lambda = G^{\lambda,+}/\text{rad } G^{\lambda,+}$  is the reductive Levi factor of  $G^{\lambda,+}$  containing  $T$ .) Let  $\mathcal{W} = N(T)/T$  be the Weyl group of  $G$ , and let  $\mathcal{W}_{G^\lambda} \subset \mathcal{W}$  be the Weyl group of  $G^\lambda$ . We write  $\bar{\rho}_\lambda \in X(T)_\mathbb{R}$  for half the sum of the positive roots of  $G^\lambda$  and  $X(T)^\lambda$  for the  $G^\lambda$ -dominant weights inside  $X(T)$ . For  $\chi \in X(T)^\lambda$  we write  $V_{G^\lambda}(\chi) = \text{Ind}_{G^\lambda \cap B}^{G^\lambda} \chi$ . I.e  $V_{G^\lambda}(\lambda)$  is the irreducible  $G^\lambda$ -representation with highest weight  $\chi$  (or sometimes also a  $G^{\lambda,+}$ -representation with the unipotent radical acting trivially). Note that  $G^\lambda$  acts on  $W_\lambda$  (when we consider it as  $G^{\lambda,+}$  representation we let  $\text{rad } G^{\lambda,+}$  act trivially). For a  $\mathcal{W}_{G^\lambda}$ -invariant  $\nu \in X(T)_\mathbb{R}$  we put

$$\mathcal{L}_{r,\lambda,\nu} = X(T)^\lambda \cap (\nu - \bar{\rho}_\lambda + r\Sigma_\lambda^0),$$

$$U_{r,\lambda,\nu} = \bigoplus_{\mu \in \mathcal{L}_{r,\lambda,\nu}} V_{G^\lambda}(\mu),$$

$$\Lambda_{r,\lambda,\nu} = (\text{End}(U_{r,\lambda,\nu}) \otimes \text{Sym}(W_\lambda))^{G^\lambda}.$$

**Proposition 1.2.1.** [ŠVdB15, Theorem 1.4.1] *Assume  $r \geq 1$ . Then one has  $\text{gldim } \Lambda_{r,\lambda,\nu} < \infty$ .*

*Proof.* As  $\nu$  is  $\mathcal{W}_{G^\lambda}$ -invariant, this follows from [ŠVdB15, Theorem 1.4.1] in the case  $\Sigma_\lambda^0 = \Sigma_\lambda$ . It is easy to modify the proof to hold also if  $\Sigma_\lambda^0 \subsetneq \Sigma_\lambda$ . (To replace  $\Sigma$  in [ŠVdB15, Theorem 1.4.1] with  $\Sigma^0$  one only needs to show that  $\text{Hom}(P_\mathcal{L}, P_\chi) = 0$  if  $\chi \notin \Gamma^0$ , and this follows by the same proof as [ŠVdB15, Lemma 11.3.1].) To replace  $\Sigma_\lambda^0$  by  $r\Sigma_\lambda^0$  is also easy.  $\square$

We say that  $x \in X$  is  $T$ -stable if  $x$  has finite stabilizer and closed  $T$ -orbit. In the case  $X = \text{Spec } W$  the existence of a  $T$ -stable point is equivalent to the cone spanned by the weights  $(\beta_i)_i$  of  $W$  being equal to  $X(T)_\mathbb{R}$ .

**Proposition 1.2.2.** *Let  $\mathcal{D} = \mathcal{D}(X/G)$  and assume that  $X$  has a  $T$ -stable point. Then there exist  $r_i \geq 1$ ,  $\lambda_i \in Y(T)^-$  and  $\mathcal{W}_{G^{\lambda_i}}$ -invariant  $\nu_i \in X(T)_\mathbb{R}$  such that  $\mathcal{D} = \langle \dots, \mathcal{D}_{-2}, \mathcal{D}_{-1}, \mathcal{D}_0 \rangle$ ,  $\mathcal{D}_{-i} \cong \mathcal{D}(\Lambda_{r_i, \lambda_i, \nu_i})$ , is a semi-orthogonal decomposition of  $\mathcal{D}$ . Moreover we may assume  $\mathcal{D}_0 \cong \mathcal{D}(\Lambda_{1,0,0})$ .*

**Remark 1.2.3.** The reader will note that (under suitable genericity conditions)  $\Lambda_{1,0,0}$  is the non-commutative resolution of  $X//G$  constructed in [ŠVdB15, Cor. 1.5.2].

**Remark 1.2.4.** If  $W$  is “quasi-symmetric” (the sum of the weights of  $W$  on each line through the origin is zero) then it is shown in [ŠVdB15, §1.6] that by replacing  $\Sigma$  by a polygon of roughly half the size one obtains smaller non-commutative resolutions for  $X//G$ . Under favourable conditions one may even obtain so-called non-commutative crepant resolutions (NCCRs) in this way (see also [HLS16] where, again under appropriate conditions, it is shown that these NCCRs are of geometric origin in the sense that they are derived equivalent to suitable  $X^{ss}/G$ ). Likewise if  $W$  is quasi-symmetric one may obtain a corresponding more refined semi-orthogonal decomposition of  $\mathcal{D}(X/G)$ . The details, which are combinatorial, will be described in a subsequent publication.

*Remark 1.2.5.* In Proposition 1.2.2 we assume that  $X$  has a  $T$ -stable point. This hypothesis is not very restrictive in view of §7 below. Roughly speaking if  $X$  does not have a  $T$ -stable point then one may easily obtain a semi-orthogonal decomposition of  $\mathcal{D}(X/G)$  involving a set of  $\mathcal{D}(X'/G')$  such that  $X'$  has a  $T'$ -stable point for  $T'$  a maximal torus of  $G'$ .

## 2. ACKNOWLEDGEMENT

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## 3. PRELIMINARIES

**3.1. Strongly étale morphisms.** Let  $G$  be reductive group. If  $G$  acts on an affine  $k$ -scheme  $X$  then we put  $X//G = \operatorname{Spec} k[X]^G$ . This is a special case of a “good quotient” (see §3.3 below). Let  $f : X \rightarrow Y$  be a  $G$ -equivariant morphism between affine  $G$ -schemes. Following [Lun73][MFK94, App. D] we say that  $f$  is *strongly étale* if  $X//G \rightarrow Y//G$  is étale and the induced morphism  $X \rightarrow Y \times_{Y//G} X//G$  is an isomorphism. This implies in particular that  $X \rightarrow Y$  is étale.

**Lemma 3.1.1.** *Assume that  $f : X \rightarrow Y$  is a strongly étale  $G$ -equivariant morphism of affine schemes and let  $H$  be a reductive subgroup of  $G$ . Then  $f$  is strongly étale as  $H$ -equivariant morphism.*

*Proof.* From the fact that  $H$  is reductive we easily obtain

$$k[X]^H = (k[Y] \otimes_{k[Y]^G} k[X]^G)^H = k[Y]^H \otimes_{k[Y]^G} k[X]^G.$$

Thus

$$k[Y] \otimes_{k[Y]^H} k[X]^H = k[Y] \otimes_{k[Y]^H} k[Y]^H \otimes_{k[Y]^G} k[X]^G = k[Y] \otimes_{k[Y]^G} k[X]^G = k[X].$$

□

**3.2. The Bialinicky-Birula decomposition in the affine case.** We use [Dri13] as a reference for some facts about the Bialinicky-Birula decomposition [BB73]. Let  $R$  be a commutative  $k$ -algebra equipped with a rational  $G_m$ -action  $\lambda : G_m \rightarrow \operatorname{Aut}_k(R)$ . This  $G_m$ -action induces a grading on  $\bigoplus_n R_n$  on  $R$  where  $z \in G_m$  acts on  $r \in R_n$  by  $z \cdot r = z^n r$ . Let  $I^+, I^-$  be the ideals in  $R$  respectively generated by  $(R_n)_{n>0}$  and  $(R_n)_{n<0}$  and put  $I = I^+ + I^-$ . We define  $R^\lambda := R/I$ .  $R^{\lambda,\pm} := R/I^\pm$ . Note

$$(3.1) \quad (R^{\lambda,\pm})_0 = R^\lambda.$$

If  $X = \operatorname{Spec} R$  then we also write  $X^\lambda = \operatorname{Spec} R^\lambda$ ,  $X^{\lambda,\pm} = \operatorname{Spec} R^{\lambda,\pm}$ . It follows from [Dri13, §1.3.4] that  $X^\lambda$  is the subscheme of fixed points of  $X$  and  $X^{\lambda,+}$ ,  $X^{\lambda,-}$  are respectively the attractor and repeller subschemes of  $X$ . According to [Dri13, Prop 1.4.20],  $X^\lambda$ ,  $X^{\lambda,\pm}$  are smooth if this is the case for  $X$ .

**Lemma 3.2.1.** *Assume that  $f : X \rightarrow Y$  is a strongly étale  $G_m$ -equivariant morphism of affine schemes, with the action denoted by  $\lambda$ . Then*

$$\begin{aligned} X^\lambda &= X \times_Y Y^\lambda \\ X^{\lambda,+} &= X \times_Y Y^{\lambda,+} \\ X^{\lambda,-} &= X \times_Y Y^{\lambda,-}. \end{aligned}$$

*Proof.* We have

$$k[X] = k[Y] \otimes_{k[Y]^{G_m}} k[X]^{G_m}$$

and this isomorphism is clearly compatible with the grading on both sides. Thus

$$k[X]_n = k[Y]_n \otimes_{k[Y]_0} k[X]_0.$$

The lemma now follows easily from the definitions.  $\square$

### 3.3. The Bialinicky-Birula decomposition when there is a good quotient.

We use the following definition from [Bri09] (see the discussion after Prop. 1.29 in loc. cit.).

**Definition 3.3.1.** Let  $G$  be a reductive group and let  $\pi : X \rightarrow Y$  be a  $G$ -equivariant morphism of  $k$ -schemes. Then  $\pi$  is a *good quotient* if the following holds

- (1)  $\pi$  is affine.
- (2)  $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$ .

It is easy to see that a good quotient is unique, if it exists. Therefore following tradition we will usually write  $Y = X//G$ . We have already used this notation in the case that  $X$  is affine. Note the following

**Lemma 3.3.2.** Assume that  $G$  is a reductive group acting on a  $k$ -scheme  $X$  such that  $X//G$  exists. Let  $H$  be a reductive subgroup of  $G$ . Then  $X//H$  also exists.

*Proof.* Let  $\pi : X \rightarrow X//G$  be the good quotient. It is easy to verify that  $X//H = \underline{\text{Spec}}(\pi_* \mathcal{O}_X)^H$ .  $\square$

Assume now that  $X$  is a  $k$ -scheme on which  $G_m$  acts via  $\lambda : G_m \rightarrow \text{Aut}(X)$ . Assume that a good quotient  $\pi : X \rightarrow X//G_m$  exists. If  $U \subset X//G_m$  is an open affine subvariety then we may define closed subvarieties  $\pi^{-1}(U)^\lambda, \pi^{-1}(U)^{\lambda, \pm}$  of  $\pi^{-1}(U)$  as in §3.2 and according to Lemma 3.2.1 these are compatible with restrictions for  $U' \subset U$ . Hence we may glue these closed subvarieties to obtain  $X^\lambda, X^{\lambda, \pm} \subset X$ . One may verify that  $X^\lambda, X^{\lambda, \pm}$  are still the fixed points and the attractor/repeller subschemes for  $\lambda$ .

**3.4. Good quotients and geometric invariant theory.** One way to obtain good quotients is via the machinery of geometric invariant theory [MFK94]. Let  $G$  be a reductive group and let  $X$  be a  $G$ -equivariant  $k$ -scheme which is projective over an affine scheme, equipped with a  $G$ -equivariant ample line bundle  $\mathcal{M}$ . If  $f \in \Gamma(X, \mathcal{M}^{\otimes n})^G$ ,  $n > 0$  then  $X_f := \{f \neq 0\} \subset X$  is affine and  $G$ -equivariant. The semi-stable locus in  $X$  is defined as  $X^{ss} = \bigcup_f X_f$ . This is an open subvariety of  $X$  which has a good quotient  $X^{ss}//G$  which may be obtained by gluing  $X_f//G = \text{Spec } k[X_f]^G$  for varying  $f$ . Another way to obtain  $X^{ss}//G$  is as follows: let  $\Gamma_*(X) = \bigoplus_n \Gamma(X, \mathcal{M}^{\otimes n})$ . Then  $X^{ss}//G = \text{Proj } \Gamma_*(X)^G$ .

The following result, which we will not use, gives an alternative description of  $X^{ss, \lambda}, X^{ss, \lambda, \pm}$  in the GIT setting.

**Proposition 3.4.1.** Let  $G$  be a reductive group and let  $X$  be a  $G$ -equivariant  $k$ -scheme which is projective over an affine scheme. Let  $\mathcal{M} \in \text{Pic}(X)$  be a  $G$ -equivariant ample line bundle  $X$  and let  $X^{ss} \subset X$  be the corresponding semi-stable locus.

Let  $R = \Gamma_*(X)$  and let  $\lambda$  be a one-parameter subgroup of  $G$ . Then  $\lambda$  acts on  $R$  in a way which is compatible with the grading and  $X^{ss, \lambda}, X^{ss, \lambda, \pm}$  are the closed subschemes of  $X^{ss}$  defined by the (graded) quotient rings  $R^\lambda, R^{\lambda, \pm}$  of  $R$  (see §3.2).

*Proof.* We will give the proof for  $X^{ss,\lambda,+}$ . The proofs for  $X^{ss,\lambda}$  and  $X^{ss,\lambda,-}$  are similar. For  $f \in R_n^G$ ,  $n > 0$  we have  $X_f = \text{Spec}(R_f)_0$ . By construction  $X^{ss,\lambda,+} \cap X_f = (X_f)^{\lambda,+}$ . We have to prove that  $(X_f)^{\lambda,+} = \text{Spec}((R^{\lambda,+})_f)_0$ . Since we have  $(X_f)^{\lambda,+} = \text{Spec}((R_f)_0)^{\lambda,+}$  this amounts to showing  $((R^{\lambda,+})_f)_0 = ((R_f)_0)^{\lambda,+}$ . By Lemma 3.2.1 we have  $(R^{\lambda,+})_f = (R_f)^{\lambda,+}$ . So ultimately we have to show  $(T^{\lambda,+})_0 = (T_0)^{\lambda,+}$  for  $T = R_f$ . Since  $\mathcal{M}$  is ample,  $T$  is strongly graded (i.e.  $T_a T_b = T_{a+b}$  for  $a, b \in \mathbb{Z}$ ) and then it is sufficient to invoke Lemma 3.4.2 below.  $\square$

**Lemma 3.4.2.** *Assume that  $T$  is  $\mathbb{Z}$ -graded strongly graded commutative  $k$ -algebra with a unit  $f \in T_n$ ,  $n > 0$ . Assume that  $T$  is in addition equipped with a  $G_m$ -action  $\lambda : G_m \rightarrow \text{Aut}(T)$  which is compatible with the grading such that  $f$  is  $G_m$ -invariant. Then we have  $(T^\lambda)_0 = (T_0)^\lambda$  and  $(T^{\lambda,\pm})_0 = (T_0)^{\lambda,\pm}$ .*

*Proof.* We give the proof for  $T^{\lambda,+}$ . We turn the  $G_m$ -action on  $T$  into a bigrading. I.e.  $T = \bigoplus_{n,\alpha \in \mathbb{Z}^2} T_{n\alpha}$  with the  $G_m$  action on  $T_{n\alpha}$  being given by  $z \cdot t = z^\alpha t$ .

We have

$$(T^{\lambda,+})_0 = T_0 / \left( \sum_{m,\alpha,\beta \in \mathbb{Z}, \beta > 0} T_{-m,\alpha} T_{m,\beta} \right),$$

$$(T_0)^{\lambda,+} = T_0 / \left( \sum_{\gamma,\delta \in \mathbb{Z}, \delta > 0} T_{0,\gamma} T_{0,\delta} \right).$$

Thus we have to show that  $T_{-m,\alpha} T_{m,\beta}$  for  $\beta > 0$  is contained in  $\sum_{\gamma,\delta \in \mathbb{Z}, \delta > 0} T_{0,\gamma} T_{0,\delta}$ . We may do this locally on  $\text{Spec } T_{00}$ . So now we assume that  $T_{00}$  is local. By Lemma 3.4.3 below this implies that  $T_0$  is graded local and since  $T_1$  is an invertible graded  $T_0$ -module it is graded free. In other words  $T \cong T_0[t, t^{-1}]$  where  $\deg(t) = (1, \sigma)$  for suitable  $\sigma \in \mathbb{Z}$ .

By hypothesis  $T_{n0}$  contains a unit  $f$ . Write  $f = ht^n$ . Then  $h \in T_{0,-n\sigma}$  is a unit in  $T_0$ . If  $\sigma \neq 0$  then it is easy to see that  $(T_0)^{\lambda,+} = 0$  and there is nothing to prove. So assume  $\sigma = 0$ . In that case we have  $T_{-m,\alpha} T_{m,\beta} = T_{0,\alpha} T_{0,\beta}$  and we are also done.  $\square$

**Lemma 3.4.3.** *Let  $S$  be a  $\mathbb{Z}$ -graded commutative ring such that  $S_0$  is local. Then  $S$  is graded local.*

*Proof.* We have to show that the homogeneous non-units form a graded ideal. Let  $x, y \in S_t$  be non-units. This implies  $xS_{-t} \subset m$ ,  $yS_{-t} \subset m$  where  $m$  is the maximal ideal of  $S_0$ . But then also  $(x+y)S_{-t} \subset m$ . So  $x+y$  is not a unit.  $\square$

**3.5. Good quotients and local generation.** Let  $X$  be a quasi-compact, quasi-separated  $G$ -scheme for a reductive group  $G$  such that a good quotient  $\pi : X \rightarrow X//G$  exists. It is easy to see that then  $X//G$  is quasi-compact and quasi-separated as well. Below we write  $\pi_s : X/G \rightarrow X//G$  for the corresponding stack morphism. Note that both  $\pi_*$  and  $\pi_{s*}$  are exact.

We recall some properties of  $D_{\text{Qch}}(X/G)$ .

**Theorem 3.5.1.** (1)  $D_{\text{Qch}}(X/G)$  is compactly generated.

(2) An object in  $D_{\text{Qch}}(X/G)$  is compact if and only if it is perfect. I.e. if and only if its image in  $D(X)$  is perfect.

(3) If  $X$  is separated then  $D_{\text{Qch}}(X/G) = \mathcal{D}(\text{Qch}(X/G))$ .

*Proof.* (1) follows from [HR14, Thm B]. It also follows from this result and the proof of [Nee92, Lemma 2.2] that every compact object is perfect. On the other

hand it is easy to see that in this case every perfect object is compact. This proves (2). Finally (3) follows from [HNR14, Thm 1.2].  $\square$

For an open  $U \subset X//G$  we write  $\tilde{U} = U \times_{X//G} X \subset X$ .

**Definition 3.5.2.** Let  $(E_i)_{i \in I}$  be a collection of perfect objects in  $D_{\text{Qch}}(X/G)$ . The subcategory of  $D_{\text{Qch}}(X/G)$  *locally generated* by  $(E_i)_{i \in I}$  is the full subcategory of  $D_{\text{Qch}}(X/G)$  spanned by all objects  $\mathcal{F}$  such that for every affine open  $U \subset X//G$  we have that  $\mathcal{F}|_{\tilde{U}}$  is in the smallest thick subcategory of  $D_{\text{Qch}}(\tilde{U}/G)$  containing  $(E_i|_{\tilde{U}})_i$ .

Let us say that  $F, G \in D_{\text{Qch}}(X/G)$  are *locally isomorphic* if there exists a covering  $X//G = \bigcup_{i \in I} U_i$  such that  $F|_{\tilde{U}_i} \cong G|_{\tilde{U}_i}$  for all  $i$ . It is convenient to call a subcategory of  $D_{\text{Qch}}(X/G)$  *locally closed* if it is closed under local isomorphism. A triangulated subcategory of  $\text{Perf}(X/G)$  is locally closed if and only if it is locally generated by itself.

**Lemma 3.5.3.** Let  $(E_i)_{i \in I}$  be a collection of perfect objects in  $D_{\text{Qch}}(X/G)$  and let  $\mathcal{F} \in \text{Perf}(X/G)$ . Let  $X//G = \bigcup_{j=1}^n U_j$  be a finite open affine covering of  $X//G$ . If for all  $j$  one has that  $\mathcal{F}|_{\tilde{U}_j}$  is in the smallest thick subcategory of  $D_{\text{Qch}}(\tilde{U}_j/G)$  containing  $(E_i|_{\tilde{U}_j})_i$  then  $\mathcal{F}$  is in the subcategory of  $D_{\text{Qch}}(X/G)$  locally generated by  $(E_i)_{i \in I}$ .

*Proof.* Let  $U \subset X//G$  be an affine open. We have to show that  $\mathcal{F}|_{\tilde{U}}$  is in the smallest thick subcategory of  $D_{\text{Qch}}(\tilde{U}/G)$  containing by  $(E_i|_{\tilde{U}})_{i \in I}$ . By replacing  $X//G$  by  $U$  and refining the cover  $U = \bigcup_{j \in I} U \cap U_j$  to an affine one we reduce to the case that  $X//G$  is itself affine. In particular by Theorem 3.5.1(3)  $D_{\text{Qch}}(X/G)$  is the derived category of  $G$ -equivariant  $k[X]$ -modules. The affine  $U_j$  yield  $G$ -equivariant flat extensions  $k[\tilde{U}_j]$  of  $k[X]$ .

Let  $\mathcal{E}$  be the cocomplete triangulated subcategory of  $D_{\text{Qch}}(X/G)$  generated by  $(E_i)_{i \in I}$  and similarly let  $\mathcal{E}_j \subseteq D_{\text{Qch}}(\tilde{U}_j/G)$  be generated by  $(E_i|_{\tilde{U}_j})_{i \in I}$ . Then there is a unique distinguished triangle

$$\mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_1 \rightarrow$$

where  $\mathcal{F}_0 \in \mathcal{E}$  and  $\mathcal{F}_1 \in \mathcal{E}^\perp$ . Since  $U_j$  is affine it is easy to see that  $\mathcal{F}_1|_{\tilde{U}_j} \in \mathcal{E}_j^\perp$ . But by hypothesis  $\mathcal{F}|_{\tilde{U}_j} \in \mathcal{E}_j$  and thus  $\mathcal{F}_1|_{\tilde{U}_j} = 0$ . Since this is true for all  $j$  we conclude  $\mathcal{F}_1 = 0$  and hence  $\mathcal{F} \in \mathcal{E}$ . Since  $\mathcal{F}$  is compact the conclusion follows from [Nee92, Lemma 2.2].  $\square$

**Lemma 3.5.4.** The category  $\text{Perf}(X/G)$  is locally generated by  $(V \otimes_k \mathcal{O}_X)_V$  where  $V$  runs through the irreducible representations of  $G$ .

*Proof.* We may assume that  $X$  is affine and then it is clear.  $\square$

**Lemma 3.5.5.** Assume that  $\mathcal{D} \subset D_{\text{Qch}}(X/G)$  is locally generated by the perfect complex  $E$  and let  $\Lambda = \pi_{s*} R\mathcal{E}nd_{X/G}(E)$  (a sheaf of DG-algebras on  $X//G$ ). The functors

$$\mathcal{D} \rightarrow \text{Perf}(\Lambda) : F \mapsto \pi_{s*} R\mathcal{H}om_{X/G}(E, F)$$

$$\text{Perf}(\Lambda) \rightarrow \mathcal{D} : H \mapsto H \overset{L}{\otimes}_\Lambda E$$

(the second functor is computed starting from a  $K$ -flat resolution of  $H$ ) yield inverse equivalences between  $\mathcal{D}$  and  $\text{Perf}(\Lambda)$ .

*Proof.* The two functors are adjoint functors between  $D_{\text{Qch}}(X/G)$  and  $D(\Lambda)$ . The fact that they define functors between  $\mathcal{D}$  and  $\text{Perf}(\Lambda)$  can be checked locally. The fact that the unit and counit are invertible can also be checked locally.  $\square$

*Remark 3.5.6.* Note that if  $U \subset X//G$  is affine then  $\Lambda|_U$  is the sheaf of DG-algebras associated to the DG-algebra  $\text{REnd}_{\tilde{U}/G}(E)$ . We will use this routinely below.

**Lemma 3.5.7.** *Assume that  $X$  is a smooth  $k$ -scheme. Let  $E \in \mathcal{D}(X/G)$ . If  $\Lambda = \pi_{s*} \text{REnd}_{X/G}(E)$  is a sheaf of algebras of finite global dimension when restricted to affine opens in  $X//G$  then the induced fully faithful functor (see Lemma 3.5.5)*

$$I : \mathcal{D}(\Lambda) \rightarrow \mathcal{D}(X/G) : H \mapsto H \overset{L}{\otimes}_{\Lambda} E$$

*is admissible (i.e. it has a left and a right adjoint).*

*Proof.* The right adjoint to  $I$  is given  $\pi_{s*} \text{RHom}_{X/G}(E, -)$ . To construct the left adjoint note that there is a duality  $\mathcal{D}(\Lambda^\circ) \rightarrow \mathcal{D}(\Lambda)$  given by  $(-)^{\vee} := \text{RHom}_{\Lambda}(-, \Lambda)$ . One checks that the left adjoint to  $I$  is given by  $\pi_{s*} \text{RHom}_{X/G}(-, E)^{\vee}$ .  $\square$

**Proposition 3.5.8.** *Let  $I$  be a totally ordered set. Assume  $\mathcal{D} \subset \text{Perf}(X/G)$  is locally generated by a collection of locally closed subcategories  $\mathcal{D}_i \in \text{Perf}(X/G)$ . Assume  $\pi_{s*} \text{RHom}_{X/G}(\mathcal{D}_i, \mathcal{D}_j) = 0$  for  $i > j$ . Then  $\mathcal{D}$  is generated by  $(\mathcal{D}_i)_i$  and in particular we have a semi-orthogonal decomposition  $\mathcal{D} = \langle \mathcal{D}_i \mid i \in I \rangle$ .*

*Proof.* It is clear that we may first reduce to the case that  $I$  finite and then to  $|I| = 2$ . Hence we assume  $I = \{1, 2\}$ . In the same vein we may reduce to the case that the  $\mathcal{D}_i$  are locally generated by single perfect complexes  $(E_i)_{i=1,2}$ . Put  $\Lambda_i = \pi_{s*} \text{End}_{X/G}(E_i)$ .

Let  $F \in \mathcal{D}$ . Put  $F_2 = \pi_{s*} \text{RHom}_{X/G}(E_2, F) \overset{L}{\otimes}_{\Lambda_2} E_2 \in \mathcal{D}_2$  and  $F_1 = \text{cone}(F_2 \rightarrow F)$ . Then  $\pi_{s*} \text{RHom}_{X/G}(E_2, F_1) = 0$ . If  $U \subset X//G$  is affine then  $F|_{\tilde{U}}$  is in the thick subcategory of  $\text{Perf}(\tilde{U}/G)$  generated by  $E_1|_{\tilde{U}}, E_2|_{\tilde{U}}$ . It follows that  $F_1|_{\tilde{U}}$  is in the thick subcategory of  $\text{Perf}(\tilde{U}/G)$  generated by  $E_1|_{\tilde{U}}$ . Since this is true for all  $U$  we conclude that  $F_1 \in \mathcal{D}_1$ . Hence  $\mathcal{D}$  is generated by  $\mathcal{D}_1, \mathcal{D}_2$ .  $\square$

### 3.6. The Bialinicky-Birula decomposition for reductive algebraic groups.

We recall the following.

**Proposition 3.6.1.** [Spr98, Proposition 8.4.5, Exercise 8.4.6(5), Theorem 13.4.2] *Let  $G$  be a connected reductive algebraic group and let  $\lambda : G_m \rightarrow G$  be a one-parameter subgroup of  $G$ . Then  $G^\lambda, G^{\lambda, \pm}$  are connected subgroups of  $G$ . Moreover the  $G^{\lambda, \pm}$  are parabolic subgroups of  $G$  and  $G^\lambda$  is the Levi-subgroup of  $G^{\lambda, \pm}$ .*

We recall the following.

**Lemma 3.6.2.** *Let  $G$  be a connected reductive algebraic group with  $T \subset B \subset G$  being a maximal torus and a Borel subgroup of  $G$ . Let  $\lambda \in Y(T)^-$  and  $\chi \in X(T)^+$  and let  $V(\chi)$  be the irreducible  $G$ -representation with highest weight  $\chi$ . Then  $\text{Res}_{G^\lambda}^G V(\chi) = V_{G^\lambda}(\chi) \oplus \bigoplus_i V_{G^\lambda}(\mu_i)$  with  $\langle \lambda, \mu_i \rangle > \langle \lambda, \chi \rangle$ .*

*Proof.* This is similar to the proof that  $\chi$  occurs with multiplicity one among the weights of  $V(\chi)$  [Jan87, Proposition 2.4]. All the weights  $\mu$  of  $V(\chi)$  satisfy  $\langle \lambda, \mu \rangle \geq \langle \lambda, \chi \rangle$ . Hence we have a decomposition  $V(\chi) = V(\chi)^\lambda \oplus V(\chi)^+$  where  $V(\chi)^+$  is the span of the weight vectors with weights  $\mu$  such that  $\langle \lambda, \mu \rangle > \langle \lambda, \chi \rangle$ . It is clear that this is a decomposition as  $G^\lambda$ -modules. If  $V(\chi)^\lambda$  is decomposable then it is easy



to see that its indecomposable summands generate distinct  $G$ -subrepresentations of  $V(\chi)$  which is impossible.

Since  $V(\chi)^\lambda$  contains the weight vector with weight  $\chi$  we must have  $V(\chi)^\lambda = V_{G^\lambda}(\chi)$ .  $\square$

**3.7. The  $G/G_e$ -action on weights.** Let  $G$  be a reductive group such that  $T \subset B \subset G_e$  are respectively a maximal torus and a Borel subgroup of  $G_e$ .

Let  $g \in G$  and  $\sigma_g = g \cdot g^{-1} \in \text{Aut}(G_e)$ . Then  $\sigma_g(T) \subset \sigma_g(B)$  are respectively a maximal torus and a Borel subgroup of  $G_e$ . Thus there exists  $g_0 \in G_e$  such that  $g_0 \sigma_g(T) g_0^{-1} = T$ ,  $g_0 \sigma_g(B) g_0^{-1} = B$ .

In the sequel if  $\bar{g} \in G/G_e$  then we write  $\sigma_{\bar{g}} \in \text{Aut}(G_e)$  for  $\sigma_{g_0 g}$  where  $g_0 g$  is an element of the coset  $\bar{g}$  such that  $\sigma_{g_0 g}$  preserves  $(T, B)$ . Since  $g_0$  is unique up to multiplication by an element of  $T$ ,  $\sigma_{\bar{g}}$  is well defined up to conjugation by an element of  $T$ . Since  $\sigma_{\bar{g}}$  preserves  $(T, B)$  it yields a well defined action on  $X(T)$  via  $\chi \mapsto \chi \circ \sigma_{\bar{g}}$  which preserves  $X(T)_{\mathbb{R}}$  and  $X(T)^+$ . We will write  $\bar{g}(\chi)$  for  $\chi \circ \sigma_{\bar{g}}^{-1}$ . There is also an action of  $G/G_e$  on  $Y(T)$  given by  $\lambda \mapsto \sigma_{\bar{g}} \circ \lambda$ . Finally we have

$$\langle \lambda, \chi \circ \sigma_{\bar{g}} \rangle = \langle \sigma_{\bar{g}} \circ \lambda, \chi \rangle.$$

If  $\lambda \in Y(T)$  then we will write  $(G/G_e)^\lambda \subset G/G_e$  for the stabilizer of  $\lambda$  under the  $G/G_e$ -action on  $Y(T)$ . Let  $\tilde{G}^\lambda$  be the inverse image of  $(G/G_e)^\lambda$  in  $G$ . There is an obvious inclusion  $(G/G_e)^\lambda \subset G_e G^\lambda / G_e = G^\lambda / G_e^\lambda$ . We will write  $\bar{G}^\lambda$  for the (open) subgroup of  $G^\lambda$  such that  $G_e^\lambda \subset \bar{G}^\lambda$  and  $\bar{G}^\lambda / G_e^\lambda = (G/G_e)^\lambda$ . So  $\tilde{G}^\lambda / G_e = \bar{G}^\lambda / G_e^\lambda$ .

For  $\chi \in X(T)^+$  we put  $V_G(\chi) := \text{Ind}_B^G \chi$ . Note that if  $G$  is not connected then  $V_G(\chi)$  will usually not be simple. We have

$$(3.2) \quad \text{Res}_{G_e}^G V_G(\chi) = \bigoplus_{\bar{g} \in G/G_e} \sigma_{\bar{g}} V_{G_e}(\chi)$$

and

$$(3.3) \quad \sigma_{\bar{g}} V_{G_e}(\chi) \cong V_{G_e}(\chi \circ \sigma_{\bar{g}}).$$

#### 4. REDUCTION SETTINGS

Now we introduce our main technical tool to obtain semi-orthogonal decompositions of  $\mathcal{D}(X/G)$ .

**4.1. Definition.** Let  $G$  be a reductive group such that  $T \subset B \subset G_e$  are respectively a maximal torus and a Borel subgroup of  $G_e$ .

Below we consider the situation where  $G$  acts on a variety  $X$ . In that case we also put for  $\chi \in X(T)^+$

$$P_\chi = V_G(\chi) \otimes_k \mathcal{O}_X \in \text{coh}(X/G).$$

To indicate context we may also write  $P_{G,\chi}$ ,  $P_{G,X,\chi}$ , etc. . . . If  $\mathcal{L} \subset X(T)^+$  then we put  $P_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} P_\chi$ . We make the following definition (using some notation introduced in §1.2).

**Definition 4.1.1.** A *reduction setting* is a tuple  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  with the following properties:

- (1)  $G$  is a reductive group and  $T \subset B \subset G_e$  are respectively a maximal torus and a Borel subgroup of  $G$ .
- (2)  $\chi \in X(T)^+$ .

- (3)  $\lambda \in Y(T)^-$ .
- (4)  $\mathcal{L}$  is a finite subset of  $X(T)^+$  invariant under  $G/G_e$ .
- (5) If  $\mu \in \mathcal{L}$  then  $\langle \lambda, \chi \rangle < \langle \lambda, \mu \rangle$ .
- (6)  $X$  is a smooth  $G$ -equivariant  $k$ -scheme such that a good quotient  $\pi : X \rightarrow X//G$  exists (with associated stack morphism  $\pi_s : X/G \rightarrow X//G$ ).
- (7) We will show in Lemma 4.1.2 below that (5) implies

$$(4.1) \quad \pi_{s*} \mathcal{R}Hom_{X/G}(P_{G,\mathcal{L}}, \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_* \mathcal{O}_{X^{\lambda,+}})) = 0$$

where  $j$  is the inclusion  $X^{\lambda,+} \hookrightarrow X$ . Consider the map

$$(4.2) \quad P_{G,\mathcal{X}} = \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes \mathcal{O}_X) \rightarrow \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_* \mathcal{O}_{X^{\lambda,+}})$$

obtained by applying  $\mathrm{RInd}_{G_e^\lambda}^G(V_{G_e^\lambda}(\chi) \otimes -)$  to the obvious map  $\mathcal{O}_X \rightarrow j_* \mathcal{O}_{X^{\lambda,+}}$ . Combining (4.2) with (4.1) we obtain a map

$$(4.3) \quad \mathrm{cone} \left( \pi_{s*} \mathcal{H}om_{X/G}(P_{G,\mathcal{L}}, P_{G,\mathcal{X}}) \overset{L}{\otimes}_{\pi_{s*} \mathcal{E}nd_{X/G}(P_{G,\mathcal{L}})} P_{G,\mathcal{L}} \rightarrow P_{G,\mathcal{X}} \right) \rightarrow \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_* \mathcal{O}_{X^{\lambda,+}})$$

We require that (4.3) is an isomorphism.

The following lemma was necessary to complete Definition 4.1.1.

**Lemma 4.1.2.** *Assume that  $\mathcal{L}$  is as in Definition 4.1.1(5). Then (4.1) holds.*

*Proof.* By using an affine covering of  $X//G$  we may assume that  $X$  is affine. By adjointness we have

$$(4.4) \quad \mathcal{R}Hom_{X/G}(P_{G,\mathcal{L}}, \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_* \mathcal{O}_{X^{\lambda,+}})) \\ = \bigoplus_{\mu \in \mathcal{L}} \mathrm{Hom}_{G_e}(\mathrm{Res}_{G_e}^G \mathrm{Ind}_{G_e}^G V_{G_e}(\mu), \mathrm{RInd}_{G_e^{\lambda,+}}^{G_e}(V_{G_e^\lambda}(\chi) \otimes_k k[X^{\lambda,+}])).$$

By the  $G/G_e$ -invariance of  $\mathcal{L}$ , the simple summands of  $(\mathrm{Res}_{G_e}^G \mathrm{Ind}_{G_e}^G V_{G_e}(\mu))_{\mu \in \mathcal{L}}$  are precisely the  $(V_{G_e}(\mu))_{\mu \in \mathcal{L}}$  (see (3.2, 3.3)). In other words it suffices to prove that for every  $\mu$  such that  $\langle \lambda, \mu \rangle > \langle \lambda, \chi \rangle$  one has

$$\mathrm{Hom}_{G_e}(V_{G_e}(\mu), \mathrm{RInd}_{G_e^{\lambda,+}}^{G_e}(V_{G_e^\lambda}(\chi) \otimes_k k[X^{\lambda,+}])) = 0$$

Note

$$(4.5) \quad \mathrm{RInd}_{G_e^{\lambda,+}}^{G_e}(V_{G_e^\lambda}(\chi) \otimes_k k[X^{\lambda,+}])) = \mathrm{RInd}_{G_e^{\lambda,+}}^{G_e}(\mathrm{RInd}_B^{G_e^{\lambda,+}}(\chi) \otimes_k k[X^{\lambda,+}])) \\ = \mathrm{RInd}_{G_e^{\lambda,+}}^{G_e} \mathrm{RInd}_B^{G_e^{\lambda,+}}(\chi \otimes_k k[X^{\lambda,+}])) \\ = \mathrm{RInd}_B^{G_e}(\chi \otimes_k k[X^{\lambda,+}])).$$

Using the fact that the weights  $\mu$  of  $k[X^{\lambda,+}]$  all satisfy  $\langle \lambda, \mu \rangle \leq 0$  (see §3.2) we conclude as in the proof of [ŠVdB15, Lemma 11.2.1] that the cohomology of  $\mathrm{RInd}_B^{G_e}(\chi \otimes_k k[X^{\lambda,+}]))$  are direct sums of  $V_{G_e}(\mu)$  with  $\langle \lambda, \mu \rangle \leq \langle \lambda, \chi \rangle$ . This finishes the proof.  $\square$

*Remark 4.1.3.* Assume that  $X$  is affine. It is easy to see that (4.3) is an isomorphism if and only if there is a complex  $P^\bullet$  in  $\mathrm{coh}(X/G)$  concentrated in degrees  $\leq 0$  with  $P^i$  being a direct sum of  $P_{G,\mu}$ ,  $\mu \in \mathcal{L}$  together with a morphism  $P^\bullet \rightarrow P_{G,\mathcal{X}}$  and a quasi-isomorphism

$$(4.6) \quad \mathrm{cone}(P^\bullet \rightarrow P_{G,\mathcal{X}}) \rightarrow \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_* \mathcal{O}_{X^{\lambda,+}})$$

which induces the canonical morphism

$$P_{G,\chi} \rightarrow \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_*\mathcal{O}_{X^{\lambda,+}})$$

after composing with  $P_{G,\chi} \rightarrow \mathrm{cone}(P^\bullet \rightarrow P_{G,\chi})$ .

From this remark we obtain the following convenient fact:

**Proposition 4.1.4.** *Assume that  $G$  is a reductive group acting on an affine variety  $X$ . If  $(G_e, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting then so is  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$ .*

*Proof.* We only have to verify (4.3) and to do so we may assume that  $X$  is affine. Assume that we have a complex and an isomorphism like (4.6) for  $G = G_e$ . Then applying the exact functor  $\mathrm{Ind}_{G_e}^G$  yields what we want.  $\square$

Note also the following

**Proposition 4.1.5.** *Assume that  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting. Then*

$$(4.7) \quad R^i \mathrm{Ind}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes j_*\mathcal{O}_{X^{\lambda,+}}) = 0 \quad \text{for } i > 0.$$

*Proof.* We assume that  $X$  is affine. Replacing  $\mathrm{Hom}_{G,X}(P_{G,\mathcal{L}}, P_{G,\chi})$  in (4.3) by its projective resolution over  $\mathrm{End}_{G,X}(P_{G,\mathcal{L}})$ , it follows that there is a nonzero cohomology only in negative degree.  $\square$

**4.2. Reduction to closed subschemes.** We will create reduction settings first in the linear case ( $X$  being a representation) and then we will restrict them to closed subschemes. To do this will use the following theorem:

**Theorem 4.2.1.** *Let  $G$  be a reductive group and let  $Y \subset X$  be a closed embedding of smooth  $G$ -varieties. If  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting then so is  $(G, B, T, Y, \mathcal{L}, \chi, \lambda)$ .*

To prove this we may assume that  $X$  is affine. We first discuss a special case.

**Lemma 4.2.2.** *Assume that  $(G, X, Y)$  are as in the statement of Theorem 4.2.1 but with  $X$  affine. Assume in addition that there is a  $G_m$ -action on  $X$  in a way which commutes with the  $G$ -action such that  $k[X]$  has only weights  $\geq 0$  and<sup>1</sup>  $k[Y] = k[X]^{G_m}$ . Then the conclusion of Theorem 4.2.1 holds.*

*Proof.* Assume that  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting. Then there exist a resolution

$$(4.8) \quad \mathrm{cone}(P_X^\bullet \rightarrow P_{G,X,\chi}) \rightarrow \mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes k[X^{\lambda,+}])$$

as in (4.6) (we have switched to coordinate ring notation). Moreover we may assume that this resolution is  $G_m$ -equivariant. In addition since  $k[X^{\lambda,+}]$  is a quotient of  $k[X]$  it only has  $G_m$ -weights  $\geq 0$  and this property is not affected by applying  $\mathrm{RInd}_{G_e^{\lambda,+}}^G(V_{G_e^\lambda}(\chi) \otimes -)$ . We conclude that as  $G \times G_m$ -equivariant  $k[X]$ -module  $P_X^n$  may be assumed to be a direct sum of  $P_{G,X,\mu} \otimes \sigma_n$ ,  $n \geq 0$  where  $\sigma_n$  is the  $G_m$ -character  $z \mapsto z^n$  and  $\mu \in \mathcal{L}$ . We then have

$$(P_{G,X,\mu} \otimes \sigma_n)^{G_m} = \begin{cases} P_{G,Y,\mu} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

---

<sup>1</sup>If the  $G_m$ -action is denoted by  $\gamma$  then this condition may also be written as  $Y = X^\gamma$ ,  $X = X^{\gamma,-}$ .

Taking  $G_m$ -invariants of (4.8) we now get a similar resolution

$$\text{cone}(P_Y^\bullet \rightarrow P_{G,Y,\chi}^\bullet) \rightarrow \text{RInd}_{G_e^{\lambda,+}}^G (V_{G_e^\lambda}(\chi) \otimes k[X^{\lambda,+}])^{G_m}.$$

Furthermore since the  $G$  and  $G_m$ -action do not interfere with each other we have

$$\text{RInd}_{G_e^{\lambda,+}}^G (V_{G_e^\lambda}(\chi) \otimes k[X^{\lambda,+}])^{G_m} = \text{RInd}_{G_e^{\lambda,+}}^G (V_{G_e^\lambda}(\chi) \otimes k[X^{\lambda,+}]^{G_m})$$

and finally using the description of  $k[X^{\lambda,+}]$  in §3.2 we easily see that  $k[X^{\lambda,+}]^{G_m} = k[Y^{\lambda,+}]$ . This finishes the proof.  $\square$

We will now reduce the proof of Theorem 4.2.1 to the special case considered in Lemma 4.2.2 using the Luna slice theorem.

**Lemma 4.2.3.** *Let  $G$  be a linear algebraic group acting on an affine variety  $X$  and let  $P \in D_{\text{Qch}}(X/G)$ . Assume that  $P$  is zero in the neighborhood of any point with closed orbit. I.e. any  $x \in X$  such that  $Gx$  is closed has an open neighborhood  $U_x$  such that  $P|_{U_x} = 0$ . Then  $P = 0$ .*

*Proof.* Put  $U' = \bigcup_x U_x$  and  $U = GU'$ . Then  $P|_U = 0$  so it is sufficient to prove  $U = X$ . Assume this is not the case. Since  $X - U$  is closed and  $G$ -invariant it contains a closed orbit (e.g. an orbit of minimal dimension). This is an obvious contradiction.  $\square$

**Lemma 4.2.4.** *If  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is such that the conditions (1-6) from Definition 4.1.1 hold, and such that  $X$  is affine, then write  $C_X$  for the cone of (4.3). If  $\alpha : Z \rightarrow X$  is a strongly étale  $G$ -equivariant morphism then  $\alpha^*(C_X) = C_Z$ .*

*Proof.* This ultimately boils down to  $\alpha^*\mathcal{O}_{X^{\lambda,+}} = \mathcal{O}_{Z^{\lambda,+}}$  which is true thanks to Lemma 3.2.1.  $\square$

*Proof of Theorem 4.2.1.* We assume that  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting and  $X$  is affine. Thus  $C_X = 0$  and we have to deduce from it  $C_Y = 0$ . According to Lemma 4.2.3 it suffices to do this in the neighborhood of any closed orbit. So let  $Gy$  be a closed orbit in  $Y$  and let  $N_Y$  be a  $G_y$ -invariant complement to  $T_y(Gy)$  in  $T_y(Y)$ . Then according to the Luna slice theorem [Lun73] there is an affine  $G_y$ -invariant “slice”  $y \in S \subset Y$  to the orbit of  $y$  and strongly  $G_y$ -equivariant étale morphism  $S \rightarrow N_Y$  which sends  $y$  to 0 such that the induced maps

$$Y \xleftarrow{\alpha} G \times^{G_y} S \xrightarrow{\beta} G \times^{G_y} N_Y$$

are strongly étale. Then by Lemma 4.2.4 we have  $\alpha^*(C_Y) = \beta^*(C_{G \times^{G_y} N_Y})$ . Thus it is sufficient to prove that  $C_{G \times^{G_y} N_Y} = 0$ .

By assumption  $Gy$  is closed in  $X$ . Let  $V$  be a  $G_y$ -invariant complement to  $T_y(Y)$  in  $T_y(X)$  and put  $N_X := N_Y \oplus V$ . By assumption  $Gy$  is closed in  $X$  and hence by the same reasoning as above we conclude that  $C_{G \times^{G_y} N_X}$  is zero in a neighborhood of the zero section of  $G \times^{G_y} N_X \rightarrow G/G_y$ . However note that  $C_{G \times^{G_y} N_X}$  is equivariant for the scalar  $G_m$ -action on  $N_X$ . So in fact  $C_{G \times^{G_y} N_X} = 0$ .

Now let  $G_m$ -act on  $N_X = N_Y \oplus V$  by acting trivially on  $N_Y$  and with weight  $-1$  on  $V$ . Then the inclusion  $G \times^{G_y} N_Y \hookrightarrow G \times^{G_y} N_X$  falls under the setting considered in Lemma 4.2.2. We conclude from this lemma that  $C_{G \times^{G_y} N_Y} = 0$ , finishing the proof.  $\square$

**4.3. Reduction settings and  $\mathcal{R}\mathcal{H}om$ .** We will prove the following result

**Lemma 4.3.1.** *Assume that we have a reduction setting  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  and assume  $\chi' \in X(T)^+$  is such that  $\langle \lambda, \chi \rangle = \langle \lambda, \chi' \rangle$  and  $\langle \lambda, \bar{g}(\chi') \rangle > \langle \lambda, \chi \rangle$  for all  $\bar{g} \notin (G/G_e)^\lambda$ . Let  $i_\lambda : X^\lambda // \bar{G}^\lambda \rightarrow X // G$  be induced from  $X^\lambda \rightarrow X$ . It is clear that  $i_\lambda$  is affine so  $i_{\lambda*}$  is exact. Let  $\pi_{s,\lambda}$  be the canonical map  $X^\lambda // \bar{G}^\lambda \rightarrow X^\lambda // \bar{G}^\lambda$ .*

*We have isomorphisms*

$$(4.9) \quad \pi_{s*} \mathcal{R}\mathcal{H}om_{X/G}(\mathrm{RInd}_{G_e^\lambda, +}^G (V_{G_e^\lambda}(\chi) \otimes \mathcal{O}_{X^\lambda, +}), \mathrm{RInd}_{G_e^\lambda, +}^G (V_{G_e^\lambda}(\chi') \otimes \mathcal{O}_{X^\lambda, +})) \\ \cong i_{\lambda*} \pi_{s,\lambda*} \mathcal{R}\mathcal{H}om_{X^\lambda // \bar{G}^\lambda}(\mathrm{Ind}_{G_e^\lambda}^{\bar{G}^\lambda} V_{G_e^\lambda}(\chi) \otimes \mathcal{O}_{X^\lambda}, \mathrm{Ind}_{G_e^\lambda}^{\bar{G}^\lambda} V_{G_e^\lambda}(\chi') \otimes \mathcal{O}_{X^\lambda}).$$

*Moreover such isomorphisms are compatible with composition when applicable.*

*Proof.* The righthand side of (4.9) only has cohomology in degree zero. Hence it is sufficient to construct an isomorphism like (4.9) in the affine case, in a way which is compatible with restriction. So we now assume  $X$  is affine. Using (4.1) and (4.3) and some applications of adjointness we have to construct an isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_{G_e}^G V_{G_e}(\chi), \mathrm{RInd}_{G_e^\lambda, +}^G (V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}])) \\ \cong \mathrm{Hom}_{\bar{G}^\lambda}(\mathrm{Ind}_{G_e^\lambda}^{\bar{G}^\lambda} V_{G_e^\lambda}(\chi), \mathrm{Ind}_{G_e^\lambda}^{\bar{G}^\lambda} V_{G_e^\lambda}(\chi') \otimes k[X^\lambda]).$$

We do this next. We have

$$(4.10) \quad \mathrm{Hom}_G(\mathrm{Ind}_{G_e}^G V_{G_e}(\chi), \mathrm{RInd}_{G_e^\lambda, +}^G (V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}])) \\ \cong \mathrm{Hom}_{G_e^\lambda, +}(\mathrm{Res}_{G_e^\lambda, +}^{G_e} \mathrm{Res}_{G_e}^G \mathrm{Ind}_{G_e}^G V_{G_e}(\chi), V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}]) \\ \stackrel{(3.2)}{\cong} \mathrm{Hom}_{G_e^\lambda, +}(\mathrm{Res}_{G_e^\lambda, +}^{G_e} \bigoplus_{\bar{g} \in G/G_e} \sigma_{\bar{g}} V_{G_e}(\chi), V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}]) \\ \cong \bigoplus_{\bar{g} \in \tilde{G}^\lambda / G_e} \mathrm{Hom}_{G_e^\lambda, +}(V_{G_e}(\chi \circ \sigma_{\bar{g}}), V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}]),$$

where the last isomorphism follows by (3.3), the assumption  $\langle \lambda, \bar{g}(\chi) \rangle > \langle \lambda, \chi' \rangle$  for all  $\bar{g} \notin (G/G_e)^\lambda = \tilde{G}^\lambda / G_e$  and the fact that all the weights  $\mu$  of  $k[X^{\lambda, +}]$  satisfy  $\langle \lambda, \mu \rangle \leq 0$  by §3.2.

Assume  $\bar{g} \in (G/G_e)^\lambda = \tilde{G}^\lambda / G_e$ . By Lemma 3.6.2, as a  $G_e^\lambda$ -representation  $V_{G_e}(\chi \circ \sigma_{\bar{g}})$  is a direct sum of  $V_{G_e^\lambda}(\chi \circ \sigma_{\bar{g}})$  and representations of the form  $V_{G_e^\lambda}(\mu)$  with  $\langle \lambda, \mu \rangle > \langle \lambda, \chi \circ \sigma_{\bar{g}} \rangle = \langle \lambda, \chi \rangle$ . For similar reasons as above such  $V_{G_e^\lambda}(\mu)$  cannot contribute to the righthand side of (4.10) so that we get

$$(4.11) \quad \mathrm{Hom}_G(\mathrm{Ind}_{G_e}^G V_{G_e}(\chi), \mathrm{Ind}_{G_e^\lambda, +}^G (V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}])) \\ \cong \bigoplus_{\bar{g} \in \tilde{G}^\lambda / G_e} \mathrm{Hom}_{G_e^\lambda, +}(V_{G_e^\lambda}(\chi \circ \sigma_{\bar{g}}), V_{G_e^\lambda}(\chi') \otimes k[X^{\lambda, +}]) \\ \cong \bigoplus_{\bar{g} \in \tilde{G}^\lambda / G_e} \mathrm{Hom}_{G_e^\lambda}(V_{G_e^\lambda}(\chi \circ \sigma_{\bar{g}}), V_{G_e^\lambda}(\chi') \otimes k[X^\lambda])$$

where the second isomorphism follows again by considering  $\lambda$ -weights and (3.1). To finish the proof we recall that by definition  $\tilde{G}^\lambda/G_e = \bar{G}^\lambda/G_e^\lambda$  and by (3.2)

$$\bigoplus_{\bar{g} \in \bar{G}^\lambda/G_e^\lambda} V_{G_e^\lambda}(\chi \circ \sigma_{\bar{g}}) = \text{Res}_{G_e^\lambda}^{\bar{G}^\lambda} \text{Ind}_{G_e^\lambda}^{\bar{G}^\lambda} V_{G_e^\lambda}(\chi).$$

It now suffice to apply the adjunction  $(\text{Res}_{G_e^\lambda}^{\bar{G}^\lambda}, \text{Ind}_{G_e^\lambda}^{\bar{G}^\lambda})$  to the righthand side of (4.11).

The compatibility with composition is a straightforward but tedious verification.  $\square$

**4.4. Reduction settings in the connected linear case.** We use the notation and conventions introduced in §1.2. We need the twisted Weyl group action of  $\mathcal{W}$  on  $X(T)$ :  $w*\chi := w(\chi + \bar{\rho}) - \bar{\rho}$ . If  $\chi \in X(T)$  and there is some  $w*\chi$  which is dominant then we write  $\chi^+ = w*\chi$ . Otherwise  $\chi^+$  is undefined.

**Proposition 4.4.1.** *Let  $G$  be a connected reductive group and assume  $B, T, \chi, \lambda, \mathcal{L}$  satisfy (1)-(5) in Definition 4.1.1. Let  $X = W^\vee$  where  $W$  is a  $G$ -representation with weights  $\beta_1, \dots, \beta_d$ . Assume  $(\chi + \beta_{i_1} + \dots + \beta_{i_{-p}})^+ \in \mathcal{L}$  for all  $\emptyset \neq \{i_1, \dots, i_{-p}\} \subseteq \{1, \dots, d\}$ ,  $i_j \neq i_{j'}$  for  $j \neq j'$ . Then  $(G, B, T, X, \mathcal{L}, \chi, \lambda)$  is a reduction setting.*

*Proof.* We only have to verify (4.3). We denote by  $K_\lambda$  the subspace of  $W$  spanned by the weight vectors  $w_j$  such that  $\langle \lambda, \beta_j \rangle > 0$ . Note that  $\text{Spec Sym}(W/K_\lambda) \cong X^{\lambda,+}$ . In [ŠVdB15, (11.3)] we constructed a quasi-isomorphism

$$(4.12) \quad C_{\lambda, \chi} \rightarrow \text{RInd}_{G^{\lambda,+}}^G (V_{G^\lambda}(\chi) \otimes k[X^{\lambda,+}]),$$

where  $C_{\lambda, \chi}$  is a complex of the form

$$(4.13) \quad C_{\lambda, \chi} \stackrel{\text{def}}{=} \left( \bigoplus_{p \leq 0, q \geq 0} H^q(G/B, (\chi \otimes_k \wedge^{-p} K_\lambda)^\vee) \otimes_k k[X][{-p-q}, d] \right).$$

Note that a priori  $C_{\lambda, \chi}$  is not concentrated in degrees  $\leq 0$  (this might be true but it would require an argument). We showed that after forgetting the differential  $C_{\lambda, \chi}$  is a sum of  $G$ -equivariant projective modules of the form  $P_\mu$  where the  $\mu$  are among the weights

$$(4.14) \quad (\chi + \beta_{i_1} + \beta_{i_2} + \dots + \beta_{i_{-p}})^+$$

(with each such expression occurring at most once) where  $\{i_1, \dots, i_{-p}\} \subset \{1, \dots, d\}$ ,  $i_j \neq i_{j'}$  for  $j \neq j'$  and  $\langle \lambda, \beta_{i_j} \rangle > 0$  [ŠVdB15, Lemma 11.2.1]. Moreover there is a single copy of  $P_\chi$  which lives in degree zero. It is not explicitly stated in loc. cit. but it follows easily from the construction that this copy of  $P_\chi$  yields an inclusion  $P_\chi \rightarrow C_{\lambda, \chi}$  such that the composition  $P_\chi \rightarrow C_{\lambda, \chi} \rightarrow \text{RInd}_{G^{\lambda,+}}^G (V_{G^\lambda}(\chi) \otimes k[X^{\lambda,+}])$  is the canonical morphism exhibited in (4.2).

Let  $C'_{\lambda, \chi} = C_{\lambda, \chi}/P_\chi$ . Then we have  $C_{\lambda, \chi} = \text{cone}(C'_{\lambda, \chi}[-1] \rightarrow P_\chi)$ . Moreover using the fact that by (4.1) we have  $\text{RHom}_{k[X]}(P_\mathcal{L}, C_{\lambda, \chi})^G = 0$  and the fact that by hypothesis the summands  $P_\mu$  of  $C'_{\lambda, \chi}$  are summands of  $P_\mathcal{L}$  we find

$$\begin{aligned} C'_{\lambda, \chi} &= \text{RHom}_{k[X]}(P_\mathcal{L}, C'_{\lambda, \chi})^G \otimes_{\text{REnd}_{k[X]}(P_\mathcal{L})^G}^L P_\mathcal{L} \\ &\cong \text{RHom}_{k[X]}(P_\mathcal{L}, P_\chi[1])^G \otimes_{\text{REnd}_{k[X]}(P_\mathcal{L})^G}^L P_\mathcal{L}. \end{aligned}$$

Hence  $C''_{\lambda, \chi}$  lives in degrees  $\leq -1$ . So we deduce that  $C_{\lambda, \chi}$  is quasi-isomorphic to a complex of the form  $\text{cone}(P^\bullet \rightarrow P_\chi)$  as in Remark 4.1.3. So (4.3) holds and we are done.  $\square$

## 5. PARTITIONING $X(T)^+$

**5.1. Preliminaries.** We assume we are in the setting of §1.2. In particular  $G$  is connected and acts on a representation  $X = W^\vee$ . We now introduce some extra notation. We let  $\Phi \subset X(T)$  be the roots of  $G$ . We write  $\Phi^-$  for the negative roots of  $G$  (the roots of  $B$ ) and  $\Phi^+$  for the positive roots. We choose a positive definite  $\mathcal{W}$ -invariant quadratic form  $(-, -)$  on  $X(T)_\mathbb{R}$ . If  $\alpha \in \Phi$  then  $\check{\alpha} \in Y(T)_\mathbb{R}$  is the corresponding coroot defined by  $\langle \check{\alpha}, \chi \rangle = 2(\alpha, \chi)/(\alpha, \alpha)$  and the associated reflection on  $X(T)_\mathbb{R}$  is defined by  $s_\alpha(\chi) = \chi - \langle \check{\alpha}, \chi \rangle \alpha$ . We put  $\Phi^\vee = \{\check{\alpha} \mid \alpha \in \Phi\}$ .

We set  $\Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda, \alpha \rangle = 0\}$ . We denote by  $\Phi_\lambda^+ = \Phi^+ \cap \Phi_\lambda$  the set of positive roots of  $G^\lambda$ .

Let us recall a slightly extended version of [ŠVdB15, Corollary D.3.] with the same proof.

**Lemma 5.1.1.** *Let  $\lambda \in Y(T)_\mathbb{R}^-$ ,  $w \in \mathcal{W}$ ,  $\chi \in X(T)$  such that  $w*\chi$  is dominant. Then  $\langle \lambda, w*\chi \rangle \leq \langle \lambda, \chi \rangle$  with equality if and only if  $w \in \mathcal{W}_{G^\lambda}$ .*

*Proof.* When comparing with [ŠVdB15, Corollary D.3.] note that in loc. cit. the role of  $\lambda$  is played by  $y$ , which is assumed to be dominant, rather than anti-dominant which is the case here. So the inequalities are reversed.

Since  $\chi^+ := w*\chi$  is dominant we have  $\chi^+ = s_{\alpha_n} * \dots * s_{\alpha_1} * \chi$  such that for each  $\chi_i := s_{\alpha_i} * \dots * s_{\alpha_1} * \chi$  the inequality  $\langle \check{\alpha}_{i+1}, \chi_i \rangle \leq -2$  holds.

In loc. cit. it is shown that  $\langle \lambda, w*\chi \rangle \leq \langle \lambda, \chi \rangle$ . Going through the proof we see that the only possibility for equality to occur is when  $\langle \lambda, \alpha_i \rangle = 0$  for all  $i$ . But then  $w' := s_{\alpha_n} \dots s_{\alpha_1} \in \mathcal{W}_{G^\lambda}$ . Since  $\chi^+$  is dominant it has trivial stabilizer for the  $*$ -action. Since  $(ww'^{-1}) * \chi^+ = \chi^+$  we conclude  $w = w' \in \mathcal{W}_{G^\lambda}$ .  $\square$

We will also need the following variant

**Lemma 5.1.2.** *Let  $\lambda \in Y(T)_\mathbb{R}^-$ ,  $w \in \mathcal{W}$ ,  $\chi \in X(T)$  such that  $w\chi$  is dominant. Then  $\langle \lambda, w\chi \rangle \leq \langle \lambda, \chi \rangle$  with equality if and only if  $w\chi \in \mathcal{W}_{G^\lambda}\chi$ .*

*Proof.* The proof is along the same lines as the proof of Lemma 5.1.1 except that  $w\chi$  may have non-trivial stabilizer. This accounts for the slightly weaker conclusion.  $\square$

We define

$$T_\lambda^+ = \{i \mid \langle \lambda, \beta_i \rangle > 0\}, \quad T_\lambda^0 = \{i \mid \langle \lambda, \beta_i \rangle = 0\}, \quad T_\lambda^- = \{i \mid \langle \lambda, \beta_i \rangle < 0\}.$$

A point  $x \in X$  is *stable* if it has closed orbit and finite stabilizer.  $X$  has a  $T$ -stable point if and only if for every  $\lambda \in Y(T) \setminus \{0\}$  there exists  $i$  such that  $\langle \lambda, \beta_i \rangle > 0$  (i.e., not all the weights lie in a half space defined by the hyperplane through the origin).

*In the rest of this section we assume that  $X = W^\vee$  has a  $T$ -stable point.*

**5.2. Expression of  $\chi$  in terms of faces of  $\Sigma$ .** As  $X$  has a  $T$ -stable point, 0 lies in the interior of the positive span of  $(\beta_i)_i$  and in particular  $-\bar{\rho} + \bigcup_r r\Sigma = X(T)_{\mathbb{R}}$ , thus every  $-\bar{\rho} \neq \chi \in X(T)$  lies either in the relative interior of a unique proper face of  $-\bar{\rho} + r\Sigma$  for a unique  $r > 0$ . We will partition the set  $X(T)^+$  according to the relative interiors of faces of  $-\bar{\rho} + r\Sigma$  to which its elements belong. However for convenience we will not use the faces directly but rather some equivalent combinatorial data associated to them.

For a set  $S$  let  $\mathcal{P}(S)$  be its power set. We put a partial ordering  $\prec$  on  $\mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3$  by declaring  $(r, S^+, S^-, S^0) \preceq (r', S'^+, S'^-, S'^0)$  if either  $r < r'$  or else  $r = r'$ ,  $S^+ \subset S'^+$  and  $S^- \subset S'^-$ .

Let  $\mathbb{R}^+ \times \mathbb{N}^3$  be equipped with the (total) lexicographic ordering. There is an order preserving map

$$(5.1) \quad (\mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3, \prec) \rightarrow (\mathbb{R}^+ \times \mathbb{N}^3, <) : (r, \mathbf{S}) \mapsto (r, |\mathbf{S}|)$$

whose fibers are incomparable among each other.

**Lemma-Definition 5.2.1.** (1) For  $-\bar{\rho} \neq \chi \in X(T)$  there exists an expression of the form

$$(5.2) \quad \chi = -\bar{\rho} - r \sum_{i \in S^+} \beta_i + 0 \sum_{i \in S^-} \beta_i + \sum_{i \in S^0} b_i \beta_i,$$

where  $r > 0$ ,  $-r < b_i < 0$ , and  $S^+, S^-, S^0$  form a partition of  $\{1, \dots, d\}$ . There is the unique tuple  $(r_\chi, \mathbf{S}_\chi) := (r_\chi, S_\chi^+, S_\chi^-, S_\chi^0)$ , for which  $(r_\chi, |\mathbf{S}_\chi|)$  is minimal among tuples attached to any such expression. If  $\chi = -\bar{\rho}$  then we put by convention  $r_\chi = 0$ ,  $S_\chi^+ = S_\chi^- = \emptyset$  (although (5.2) is then not true). Below we refer to this situation as the “trivial case”.

(2) If  $\chi \in X(T)^+$  then there exists  $\lambda \in Y(T)_{\mathbb{R}}^-$  with the properties:  $S_\chi^- = T_\lambda^-$ ,  $S_\chi^+ = T_\lambda^+$ ,  $S_\chi^0 = T_\lambda^0$ .

*Remark 5.2.2.* If  $\chi \in X(T)^+$  then the trivial case is equivalent to  $G = T$  and  $\chi = 0$ .

*Remark 5.2.3.* It will follow from the proof of Lemma 5.2.1 as well a Lemma 5.2.4 below that the data  $(r_\chi, \mathbf{S}_\chi)$  identifies which proper face of  $-\bar{\rho} + r_\chi \Sigma$  contains  $\chi$  in its relative interior. Furthermore  $\lambda$  as in Lemma 5.2.1(2) defines an appropriately chosen supporting plane for that face. Finally the  $\prec$ -ordering is opposite to the ordering given by inclusion of faces.

*Proof of Lemma 5.2.1.* The existence of an expression with minimal  $(r_\chi, |\mathbf{S}_\chi|)$  is obvious. To prove the uniqueness of the associated tuple  $(r_\chi, \mathbf{S}_\chi)$  assume that there are two minimal expressions with different associated tuples. Taking their average we obtain an expression which is smaller than both the original expressions, contradicting the minimality.

We will now prove (2). If we are in the trivial case then we take  $\lambda = 0$ . So we will now assume we are not in the trivial case. We take  $r$  minimal such that  $\chi \in -\bar{\rho} + r\Sigma$  (and hence  $r_\chi = r$ ). By [ŠVdB15, Lemma C.2] there exists  $0 \neq \lambda \in Y(T)_{\mathbb{R}}$  such that  $\langle \lambda, \chi \rangle < \langle \lambda, \mu \rangle$  for all  $\mu \in -\bar{\rho} + r_\chi \Sigma$  and  $\chi$  can be written as

$$(5.3) \quad \chi = -\bar{\rho} - r_\chi \sum_{i \in T_\lambda^+} \beta_i + \sum_{i \in T_\lambda^0} b_i \beta_i,$$

$-r_\chi < b_i < 0$ . Thus by Lemma 5.2.4 below  $\chi + \bar{\rho}$  is in the relative interior of the face of  $r_\chi \Sigma$  defined by the supporting half plane  $\langle \lambda, \chi + \bar{\rho} \rangle \leq \langle \lambda, - \rangle$ . Let  $w \in \mathcal{W}$



be such that  $w\lambda \in Y(T)_{\mathbb{R}}^-$ . By the discussion preceding the [ŠVdB15, (11.4)]  $\langle w\lambda, \chi + \bar{\rho} \rangle \leq \langle w\lambda, - \rangle$  is still a supporting half plane for  $r_{\chi}\bar{\Sigma}$ . It is easy to see that the corresponding face must be equal to  $wF$ . Since the face still contains  $\chi + \bar{\rho}$  we must have  $F = wF$ . It follow again from Lemma 5.2.4 below that (5.3) remains true with  $\lambda$  replaced by  $w\lambda$ . So we now assume  $\lambda \in Y(T)_{\mathbb{R}}^-$ .

By Observation (3) in the proof of [ŠVdB15, Theorem 1.4.1] (applied to both boundaries of the interval  $[-r_{\chi}, 0]$ ) we find that in any expression of the form (5.2) we must have  $T_{\lambda}^+ \subset S^+$ ,  $T_{\lambda}^- \subset S^-$ . Since  $(|S_{\chi}^+|, |S_{\chi}^-|)$  is minimal this implies  $S_{\chi}^{\pm} = T_{\lambda}^{\pm}$ ,  $S_{\chi}^0 = T_{\lambda}^0$ , establishing (2).  $\square$

We have used the following result.

**Lemma 5.2.4.**  $\chi \in X(T)_{\mathbb{R}}$  is in the relative interior of a face of  $\bar{\Sigma}$  defined by a supporting half plane  $\langle \lambda, \chi \rangle \leq \langle \lambda, - \rangle$  for  $\lambda \neq 0$  if and only if

$$\chi = - \sum_{i \in T_{\lambda}^+} \beta_i + \sum_{i \in T_{\lambda}^0} b_i \beta_i,$$

with  $-1 < b_i < 0$ .

*Proof.* Let  $H$  be the hyperplane  $\langle \lambda, \chi \rangle = \langle \lambda, - \rangle$ . Then  $F = H \cap \bar{\Sigma}$  is given by those  $\mu = \sum_i c_i \beta_i$  such that  $\langle \lambda, \chi \rangle = \langle \lambda, \mu \rangle$  and

$$\begin{aligned} i \in T_{\lambda}^+ &\Rightarrow c_i = 1, \\ i \in T_{\lambda}^- &\Rightarrow c_i = 0, \\ i \in T_{\lambda}^0 &\Rightarrow c_i \in [0, 1]. \end{aligned}$$

This follows in fact from Observation (3) in the proof of [ŠVdB15, Theorem 1.4.1] (applied to both boundaries of the interval  $[-1, 0]$ ). It follows that relint  $F$  is given by those  $\mu = \sum_i c_i \beta_i$  such that  $\langle \lambda, \chi \rangle = \langle \lambda, \mu \rangle$  and

$$\begin{aligned} i \in T_{\lambda}^+ &\Rightarrow c_i = 1, \\ i \in T_{\lambda}^- &\Rightarrow c_i = 0, \\ i \in T_{\lambda}^0 &\Rightarrow c_i \in ]0, 1[. \end{aligned}$$

Applying this with  $\mu = \chi$  yields what we want.  $\square$

**Lemma 5.2.5.** Assume that  $\lambda$  corresponds to  $\chi$  as in Lemma 5.2.1. Then the following properties hold

- (1)  $\langle \lambda, \chi \rangle = \langle \lambda, \chi' \rangle < \langle \lambda, \mu \rangle$  for all  $\mu \in X(T)^+$  with  $(r_{\chi}, \mathbf{S}_{\chi}) \not\leq (r_{\mu}, \mathbf{S}_{\mu})$  and all  $\chi' \in X(T)^+$  with  $(r_{\chi'}, \mathbf{S}_{\chi'}) = (r_{\chi}, \mathbf{S}_{\chi})$ ,
- (2) the sets  $\{\beta_i \mid i \in S_{\chi}^+\}$ ,  $\{\beta_i \mid i \in S_{\chi}^-\}$  are  $\mathcal{W}_{G^{\lambda}}$ -invariant.

*Proof.* (2) is immediate since  $\lambda$  is stabilized by  $\mathcal{W}_{G^{\lambda}}$ .

Now we verify (1). Again it is sufficient to consider the non-trivial case. The part involving  $\chi'$  is easy so we will discuss the part involving  $\mu$ . We have

$$\mu = -\bar{\rho} - r_{\mu} \sum_{i \in S_{\mu}^+} \beta_i + \sum_{i \in S_{\mu}^0} b_i \beta_i,$$

with  $-r_{\mu} < b_i < 0$ . Write  $\beta_i = \langle \lambda, \beta_i \rangle$ . Then

$$(5.4) \quad \langle \lambda, \chi \rangle = -\langle \lambda, \bar{\rho} \rangle - r_{\chi} \sum_{i \in T_{\chi}^+} \beta_i$$

and

$$\begin{aligned}
\langle \lambda, \mu \rangle &= -\langle \lambda, \bar{\rho} \rangle - r_\mu \sum_{i \in S_\mu^+} \mathbb{B}_i + \sum_{i \in S_\mu^0} b_i \mathbb{B}_i \\
&= -\langle \lambda, \bar{\rho} \rangle - r_\mu \sum_{i \in S_\mu^+ \cap T_\lambda^+} \mathbb{B}_i - r_\mu \sum_{i \in S_\mu^+ \cap T_\lambda^-} \mathbb{B}_i + \sum_{i \in S_\mu^0 \cap T_\lambda^+} b_i \mathbb{B}_i + \sum_{i \in S_\mu^0 \cap T_\lambda^-} b_i \mathbb{B}_i \\
&\geq -\langle \lambda, \bar{\rho} \rangle - r_\mu \sum_{i \in S_\mu^+ \cap T_\lambda^+} \mathbb{B}_i - 0 \sum_{i \in S_\mu^+ \cap T_\lambda^-} \mathbb{B}_i - r_\mu \sum_{i \in S_\mu^0 \cap T_\lambda^+} \mathbb{B}_i + 0 \sum_{i \in S_\mu^0 \cap T_\lambda^-} \mathbb{B}_i \\
&= -\langle \lambda, \bar{\rho} \rangle - r_\mu \sum_{i \in (S_\mu^+ \cup S_\mu^0) \cap T_\lambda^+} \mathbb{B}_i \\
&\geq -\langle \lambda, \bar{\rho} \rangle - r_\mu \sum_{i \in T_\lambda^+} \mathbb{B}_i.
\end{aligned}$$

The total inequality will be strict if any of the following conditions hold.

$$\begin{aligned}
S_\mu^+ \cap T_\lambda^- &\neq \emptyset & \text{or} \\
S_\mu^0 \cap (T_\lambda^+ \cup T_\lambda^-) &\neq \emptyset & \text{or} \\
(S_\mu^+ \cup S_\mu^0) &\not\supset T_\lambda^+
\end{aligned}$$

which is equivalent to any of the following conditions holding

$$\begin{aligned}
(5.5) \quad S_\mu^+ &\not\supset T_\lambda^+ & \text{or} \\
S_\mu^- &\not\supset T_\lambda^- & \text{or} \\
S_\mu^0 &\not\supset T_\lambda^0.
\end{aligned}$$

To prove (1) we have to show

$$(r_\chi, \mathbf{S}_\chi) \not\leq (r_\mu, \mathbf{S}_\mu) \Rightarrow \langle \lambda, \chi \rangle < \langle \mu, \chi \rangle.$$

The condition on the left hand side is equivalent to any of the following conditions holding

$$\begin{aligned}
(5.6) \quad r_\chi &> r_\mu & \text{or} \\
r_\chi = r_\mu &\text{ and } T_\lambda^+ \not\subset S_\mu^+ & \text{or} \\
r_\chi = r_\mu &\text{ and } T_\lambda^- \not\subset S_\mu^-.
\end{aligned}$$

(1) now follows by comparing (5.5) with (5.6). To obtain the desired result we also use  $T_\lambda^+ \neq \emptyset$  which follows from the existence of a  $T$ -stable point.  $\square$

**Corollary 5.2.6.** *Let  $\chi \in X(T)^+$  be such that  $r_\chi \geq 1$  and let  $\lambda$  be as in Lemma 5.2.1(2). If  $p > 0$  and  $\mu = \chi + \beta_{i_1} + \dots + \beta_{i_p}$ , where  $\{i_1, \dots, i_p\} \subset \{1, \dots, d\}$ ,  $i_j \neq i_{j'}$  for  $j \neq j'$  and  $\langle \lambda, \beta_{i_j} \rangle > 0$ , then  $(r_{\mu^+}, |\mathbf{S}_{\mu^+}|) < (r_\chi, |\mathbf{S}_\chi|)$ . Moreover,  $\langle \lambda, \chi \rangle < \langle \lambda, \mu^+ \rangle$ .*

*Proof.* By the property (2) in Lemma 5.2.1 every  $k$  for which  $\langle \lambda, \beta_k \rangle > 0$  belongs to  $S_\chi^+$  and thus  $r_\mu < r_\chi$  or  $r_\mu = r_\chi$  and  $|\mathbf{S}_\mu| < |\mathbf{S}_\chi|$ . In other words  $(r_\mu, |\mathbf{S}_\mu|) < (r_\chi, |\mathbf{S}_\chi|)$ .

As  $(r_\mu, |\mathbf{S}_\mu|)$  depends only on the  $\mathcal{W}$ -orbit of  $\mu$  for the  $*$ -action, we also have  $(r_{\mu^+}, |\mathbf{S}_{\mu^+}|) < (r_\chi, |\mathbf{S}_\chi|)$  and hence  $(r_{\mu^+}, \mathbf{S}_{\mu^+}) \not\leq (r_\chi, \mathbf{S}_\chi)$ . Property (1) in Lemma 5.2.5 then implies  $\langle \lambda, \mu^+ \rangle > \langle \lambda, \chi \rangle$ .  $\square$

We denote

$$\chi_p = -r_\chi \sum_{i \in S_\chi^+} \beta_i$$

The following lemma gives a description of the set of  $\chi$  with given  $(r_\chi, \mathbf{S}_\chi)$  in terms of objects related to  $G^\lambda$ .

**Lemma 5.2.7.** *Let  $\chi \in X(T)$  and let  $\lambda$  be as in Lemma 5.2.1(2). Then the set*

$$(5.7) \quad \{\chi' \in X(T)^+ \mid (r_{\chi'}, \mathbf{S}_{\chi'}) = (r_\chi, \mathbf{S}_\chi)\}$$

*is equal to*

$$(\nu - \bar{\rho}_\lambda + r_\chi \Sigma_\lambda^0) \cap X(T)^\lambda$$

*where*

$$\nu = -\bar{\rho} + \bar{\rho}_\lambda + \chi_p.$$

*Moreover  $\nu$  is  $\mathcal{W}_{G^\lambda}$ -invariant.*

*Proof.* The fact that  $\nu$  is  $\mathcal{W}_{G^\lambda}$ -invariant is a direct consequence of Lemma 5.2.5 (2) and the standard fact that  $-\bar{\rho} + \bar{\rho}_\lambda$  is  $\mathcal{W}_{G^\lambda}$ -invariant.

By Lemma 5.2.1 we have

$$\{\chi' \in X(T)^+ \mid (r_{\chi'}, \mathbf{S}_{\chi'}) = (r_\chi, \mathbf{S}_\chi)\} = (-\bar{\rho} + \chi_p + r_\chi \Sigma_\lambda^0) \cap X(T)^+.$$

By Lemma 5.2.8 below we may rewrite this as

$$\{\chi' \in X(T)^\lambda \mid (r_{\chi'}, \mathbf{S}_{\chi'}) = (r_\chi, \mathbf{S}_\chi)\} = (-\bar{\rho} + \chi_p + r_\chi \Sigma_\lambda^0) \cap X(T)^\lambda$$

which is the same as

$$(\nu - \bar{\rho}_\lambda + r_\chi \Sigma_\lambda^0) \cap X(T)^\lambda. \quad \square$$

We have used the following result.

**Lemma 5.2.8.** *Let  $\chi \in X(T)^+$  and let  $\lambda$  be as in Lemma 5.2.1(2). If  $\chi \in X(T)^+$  and  $\chi' \in X(T)^\lambda$  such that  $(r_{\chi'}, \mathbf{S}_{\chi'}) = (r_\chi, \mathbf{S}_\chi)$  then  $\chi' \in X(T)^+$ .*

*Proof.* We assume that we are not in the trivial case otherwise there is nothing to do. We first verify  $s_\alpha * \chi' \neq \chi'$  for all  $\alpha \in \mathcal{W}$ . This implies that  $\chi'^+$  exists. Assume on the contrary that  $s_\alpha * \chi' = \chi'$  for some  $\alpha$ . By the uniqueness of the minimal expression in Lemma 5.2.1 and the fact that  $(r_{\chi'}, |\mathbf{S}_{\chi'}|)$  does not depend on the  $\mathcal{W}$ -orbit for the  $*$ -action we obtain that  $S_{\chi'}^+ = S_\chi^+ = T_\lambda^+$  is  $s_\alpha$ -invariant. Using the formula (5.4) we find  $\langle \lambda, \chi \rangle = \langle \lambda, s_\alpha * \chi \rangle$ . Then Lemma 5.1.1 implies  $s_\alpha \in \mathcal{W}_{G^\lambda}$ . However this is excluded by the fact that  $s_\alpha * \chi' = \chi'$  and  $\chi' \in X(T)^\lambda$ .

Assume  $\chi' \notin X(T)^+$ . By the above discussion there exists  $1 \neq w \in \mathcal{W}$  such that  $w * \chi'$  is dominant. Furthermore  $w \notin \mathcal{W}_{G^\lambda}$  as  $\chi' \in X(T)^\lambda$ . Then Lemma 5.1.1 implies  $\langle \lambda, w * \chi' \rangle < \langle \lambda, \chi' \rangle = \langle \lambda, \chi \rangle$ .

From the fact that  $(r_{w * \chi'}, |\mathbf{S}_{w * \chi'}|) = (r_{\chi'}, |\mathbf{S}_{\chi'}|) = (r_\chi, |\mathbf{S}_\chi|)$  we deduce  $(r_\chi, \mathbf{S}_\chi) \not\leq (r_{w * \chi'}, \mathbf{S}_{w * \chi'})$ . Then property (1) in Lemma 5.2.5 implies  $\langle \lambda, \chi \rangle \leq \langle \lambda, w * \chi' \rangle$ . This is a contradiction.  $\square$

**5.3.  $G/G_e$ -action on  $X(T)^+$ .** Here we allow  $G$  to be non-connected.  $W$  is still a  $G$ -representation. We apply the previous results with  $G$  replaced by  $G_e$ . In particular  $T \subset B \subset G_e$ . As we have seen in §3.7 the group  $G/G_e$  acts on  $X(T)^+$  via  $\bar{g}(\chi) = \chi \circ \sigma_{\bar{g}}^{-1}$ .

Since  $G/G_e$  also acts on the weights on  $W$ , it may be made to act on  $\{1, \dots, d\}$  via  $\bar{g}(i) = j$  if  $\beta_j = \beta_i \circ \sigma_{\bar{g}}^{-1}$ . This action extends to an action of  $G/G_e$  on the partially ordered set  $(\mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3, \prec)$  introduced above.

**Lemma 5.3.1.** *The map*

$$(5.8) \quad X(T)^+ \mapsto \mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3 : \chi \mapsto (r_\chi, \mathbf{S}_\chi)$$

*is  $G/G_e$ -equivariant. Moreover if  $\chi \in X(T)^+$ ,  $\bar{g} \in G/G_e$  and  $\lambda \in Y(T)_{\mathbb{R}}$  is as in Lemma 5.2.1(2) then  $\sigma_{\bar{g}} \circ \lambda \in Y(T)_{\mathbb{R}}^-$  satisfies property (2) in Lemma 5.2.1 for  $\chi \circ \sigma_{\bar{g}}$ . Consequently,  $(T_{\sigma_{\bar{g}} \circ \lambda}^\pm, T_{\sigma_{\bar{g}} \circ \lambda}^0) = \bar{g}^{-1}(T_\lambda^\pm, T_\lambda^0)$ .*

*Proof.* Let  $\chi \in X(T)^+$ . As usual we may assume we are in the non-trivial case. We obtain an expression of  $\chi \circ \sigma_{\bar{g}}$  of the form (5.2) with  $\beta_i$  replaced by  $\beta_i \circ \sigma_{\bar{g}}$ . Since this expression is minimal for  $\chi \circ \sigma_{\bar{g}}$  (as otherwise applying  $- \circ \sigma_{\bar{g}}^{-1}$  would give a smaller expression for  $\chi$ ),  $(r_{\chi \circ \sigma_{\bar{g}}}, \mathbf{S}_{\chi \circ \sigma_{\bar{g}}}) = \bar{g}^{-1}(r_\chi, \mathbf{S}_\chi)$ .

As  $\langle \lambda, \beta \circ \sigma_{\bar{g}} \rangle = \langle \sigma_{\bar{g}} \circ \lambda, \beta \rangle$  for all  $\beta \in X(T)$ , it is easy to verify  $\sigma_{\bar{g}} \circ \lambda \in Y(T)_{\mathbb{R}}^-$  satisfies property (2) in Lemma 5.2.1 for  $\chi \circ \sigma_{\bar{g}}$ . This implies in particular the last assertion of the lemma.  $\square$

**5.4. Reduction settings.** Here we allow  $G$  to be again non-connected.

**Lemma 5.4.1.** *There exists a total ordering  $<$  on  $\mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3$  such that the following conditions hold.*

- (1) *If  $(r, |\mathbf{S}|) < (r', |\mathbf{S}'|)$  then  $(r, \mathbf{S}) < (r', \mathbf{S}')$ .*
- (2) *If  $(r, \mathbf{S}) < (r', \mathbf{S}')$  and  $(r, \mathbf{S})$  and  $(r', \mathbf{S}')$  are in different  $G/G_e$ -orbits then  $\bar{g}(r, \mathbf{S}) < \bar{h}(r', \mathbf{S}')$  for all  $\bar{g}, \bar{h}$  in  $G/G_e$ .*

*Proof.* The map (5.1) is  $G/G_e$ -equivariant. We choose an arbitrary totally ordering on the fibers of (5.1) compatible with condition (2). Combining this with (1) completely fixes  $<$ .  $\square$

**Remark 5.4.2.** It is clear that any total ordering  $<$  as in Lemma 5.4.1(1) refines the partial ordering  $\prec$ .

**Lemma-Definition 5.4.3.** *It is possible to choose for any  $\chi \in X(T)^+$  a  $\lambda_\chi \in Y(T)_{\mathbb{R}}^-$  such that the following conditions are satisfied*

- (1)  *$\lambda = \lambda_\chi$  satisfies the property (2) in Lemma 5.2.1.*
- (2) *If  $(r_{\chi'}, \mathbf{S}_{\chi'}) = (r_\chi, \mathbf{S}_\chi)$  then  $\lambda_\chi = \lambda_{\chi'}$ .*
- (3) *We have  $\lambda_{\chi \circ \sigma_{\bar{g}}} = \sigma_{\bar{g}} \circ \lambda_\chi$  for all  $\bar{g} \in G/G_e$ .*
- (4) *If  $\bar{g}(T_{\lambda_\chi}^\pm, T_{\lambda_\chi}^0) = (T_{\lambda_\chi}^\pm, T_{\lambda_\chi}^0)$  then  $\sigma_{\bar{g}} \circ \lambda_\chi = \lambda_\chi$ .*

*Proof.* Choose representatives  $(r_{\chi_i}, \mathbf{S}_{\chi_i})$  for the orbits of the  $G/G_e$ -action on the image of (5.8). For each  $i$  choose  $\lambda'_i \in Y(T)^-$  such that  $S_{\chi_i}^- = T_{\lambda'_i}^-$ ,  $S_{\chi_i}^+ = T_{\lambda'_i}^+$ ,  $S_{\chi_i}^0 = T_{\lambda'_i}^0$  as in Lemma 5.2.1(2). Let  $\bar{G}_i \subset G/G_e$  be the stabilizer of  $(r_{\chi_i}, \mathbf{S}_{\chi_i})$  and put  $\lambda_i = \sum_{\bar{g} \in \bar{G}_i} \sigma_{\bar{g}} \circ \lambda'_i$ . Then it is easy to see that we still have  $S_{\chi_i}^- = T_{\lambda_i}^-$ ,  $S_{\chi_i}^+ = T_{\lambda_i}^+$ ,  $S_{\chi_i}^0 = T_{\lambda_i}^0$  and moreover  $\bar{g} \circ \lambda_i = \lambda_i$  if  $\bar{g} \in \bar{G}_i$ .

Now for  $\chi \in X(T)^+$  write  $(r_\chi, \mathbf{S}_\chi) = \bar{h}(r_{\chi_i}, \mathbf{S}_{\chi_i})$  for suitable  $i$  and  $\bar{h} \in G/G_e$ . Then we put  $\lambda_\chi = \sigma_{\bar{h}} \circ \lambda_i$ . It is clear that this is well defined and has the requested properties.  $\square$

**Lemma 5.4.4.** *Let  $(\lambda_\chi)_\chi$  be as in Lemma 5.4.3. We have for  $\bar{g} \in G/G_e$ :*

$$\langle \lambda_\chi, \bar{g}(\chi) \rangle \geq \langle \lambda_\chi, \chi \rangle$$

*with equality if and only if  $\bar{g} \in (G/G_e)^{\lambda_\chi}$ .*

*Proof.* By (5.8) we have

$$(5.9) \quad (r_{\bar{g}(\chi)}, \mathbf{S}_{\bar{g}(\chi)}) = \bar{g}(r_\chi, \mathbf{S}_\chi).$$

Hence in particular

$$(r_{\bar{g}(\chi)}, |\mathbf{S}_{\bar{g}(\chi)}|) = (r_\chi, |\mathbf{S}_\chi|)$$

and so  $(r_\chi, \mathbf{S}_\chi) \not\prec (r_{\bar{g}(\chi)}, \mathbf{S}_{\bar{g}(\chi)})$ . It follows from Lemma 5.2.5(2) that

$$\langle \lambda_\chi, \chi \rangle \leq \langle \lambda_\chi, \bar{g}(\chi) \rangle.$$

Also by Lemma 5.2.5 equality will happen precisely when  $(r_{\bar{g}(\chi)}, \mathbf{S}_{\bar{g}(\chi)}) = (r_\chi, \mathbf{S}_\chi)$  which by (5.9) implies that  $\bar{g}$  stabilizes  $\mathbf{S}_\chi = (T_{\lambda_\chi}^+, T_{\lambda_\chi}^-, T_{\lambda_\chi}^0)$ . By Lemma 5.4.3(4) this implies  $\bar{g} \in (G/G_e)^{\lambda_\chi}$ .  $\square$

Below we fix  $(\lambda_\chi)_\chi$  as in Lemma 5.4.3 and we choose a total ordering on  $\mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3$  as in Lemma 5.4.1. We put the induced ordering on  $I := \{(r_\chi, \mathbf{S}_\chi) \mid \chi \in X(T)^+\} \subset \mathbb{R}^+ \times \mathcal{P}(\{1, \dots, d\})^3$ . As a totally ordered set we have  $I \cong \mathbb{N}$ . For  $i \in I$  we put  $F_i = \{\chi \in X(T)^+ \mid (r_\chi, \mathbf{S}_\chi) = i\}$ . This gives a  $G/G_e$ -equivariant partition

$$(5.10) \quad X(T)^+ = \coprod_{i \in I} F_i.$$

In each  $F_i$  we choose one representative which we denote by  $\chi_i$ . We write  $\lambda_i = \lambda_{\chi_i}$ ,  $r_i = r_{\chi_i}$ ,  $\mathbf{S}_i = \mathbf{S}_{\chi_i}$ . By our choice of  $\lambda_\chi$  and the definition of  $F_i$ ,  $(r_i, \lambda_i, \mathbf{S}_i)$  depends only on  $i \in I$  and not on the choice of  $\chi_i \in F_i$ .

For  $j \in I$  we write

$$\begin{aligned} \mathcal{L}_{<j} &= \bigcup_{i \in I, i < j} F_i \subset X(T)^+, \\ \mathcal{L}_{\leq j} &= \bigcup_{i \in I, i \leq j} F_i \subset X(T)^+. \end{aligned}$$

Let  $J \subset I$  be the minimal representatives for the orbits of the action of  $G/G_e$  on  $I$ . By the choice of  $J$  and property (2) in Lemma 5.4.1 the set  $\{i \in I \mid i < j\}$  is  $G/G_e$ -invariant if  $j \in J$  and hence the same is true for  $\mathcal{L}_{<j}$ .

**Corollary 5.4.5.** *Let  $j \in J$  be such that  $r_j \geq 1$  and  $\chi \in F_j$ . Then  $(G, B, T, X, \mathcal{L}_{<j}, \chi, \lambda_j)$  is a reduction setting.*

*Proof.* We may reduce to the case that  $X$  is affine. Then by Theorem 4.2.1 and Proposition 4.1.4 we may assume  $G = G_e$  and  $X = W^\vee$ . Thus we need to verify the assumptions of Proposition 4.4.1. They are satisfied by Corollary 5.2.6 due to the choice of  $\lambda_j = \lambda_\chi$  for every  $\chi \in F_j$ .  $\square$

## 6. PROOFS OF THE SEMI-ORTHOGONAL DECOMPOSITIONS

In this section we will prove Theorem 1.1.2 and along the way we will also prove Proposition 1.2.2 and Corollary 7.2. For the proof of Theorem 1.1.2 we select a finite open affine covering  $X//G = \bigcup_i U_i$  and put  $X_i = \pi^{-1}(U_i)$ . Thus  $X = \bigcup_i X_i$  for  $G$ -equivariant affine  $G$ -varieties  $X_i$ . We choose a  $G$ -representation  $W$  such that  $W^\vee$  has a  $T$ -stable point together with a closed  $G$ -equivariant embedding  $\coprod_i X_i \hookrightarrow W^\vee$ . We use  $W$  to construct a partition (5.10) of  $X(T)^+$  as in §5.4. The arguments below will be based on “reduction to the affine case”, i.e. to one of the  $X_i$ .

For  $j \in I$  let  $\mathcal{D}_{<j}, \mathcal{D}_{\leq j}$  be the triangulated subcategories of  $\mathcal{D}(X/G)$  locally generated by  $P_{\mathcal{L}_{<j}}, P_{\mathcal{L}_{\leq j}}$  as in §3.5. and put  $\Lambda_{<j} = \pi_{s*} \mathcal{E}nd_{X/G}(P_{\mathcal{L}_{<j}})$ .

For  $j \in J$  let  $\mathcal{D}_j$  be the triangulated subcategory of  $\mathcal{D}(X/G)$  locally generated by  $\langle \mathrm{RInd}_{G_e}^G (V_{G_e}^{\lambda_j}(\chi) \otimes \mathcal{O}_{X^{\lambda_j,+}}) \mid \chi \in F_j \rangle$ .

For a  $\mathcal{W}_{G_e}^\lambda$ -invariant  $\nu \in X(T)_\mathbb{R}$  we put

$$\begin{aligned} \mathcal{L}_{r,\lambda,\nu} &= X(T)^\lambda \cap (\nu - \bar{\rho}_\lambda + r\Sigma_\lambda^0), \\ U_{r,\lambda,\nu} &= \bigoplus_{\mu \in \mathcal{L}_{r,\lambda,\nu}} \mathrm{Ind}_{G_e}^{\bar{G}^\lambda} V_{G_e}^\lambda(\mu), \\ \Lambda_{r,\lambda,\nu} &= (\mathrm{End}(U_{r,\lambda,\nu}) \otimes \mathcal{O}_{X^\lambda})^{\bar{G}^\lambda}. \end{aligned}$$

Proposition 1.2.2 and Theorem 1.1.2 will be consequences of the following proposition.

**Proposition 6.1.** (1) If  $j \notin J$  then  $\mathcal{D}_{<j} = \mathcal{D}_{\leq j}$ .

Assume that  $j \in J$  is such that  $r_j \geq 1$ . Then

- (2)  $\mathcal{D}_{<j} \cong \mathcal{D}(\Lambda_{<j})$  and  $\Lambda_{<j}$  has finite global dimension when restricted to affine opens in  $X//G$ .
- (3)  $\mathcal{D}_j \cong \mathcal{D}(\Lambda_{r_j,\lambda_j,\nu_j})$  for  $\nu_j = -\bar{\rho} + \bar{\rho}_{\lambda_j} + (\chi_j)_p$  (where  $(\chi_j)_p$  was introduced in §5.2) and  $\Lambda_{r_j,\lambda_j,\nu_j}$  has finite global dimension when restricted to affine opens in  $X//G$ .
- (4)  $\mathcal{D}_{\leq j} = \langle \mathcal{D}_j, \mathcal{D}_{<j} \rangle$  is a semi-orthogonal decomposition of  $\mathcal{D}_{\leq j}$ .
- (5) One has  $\mathcal{D}(X/G) = \bigcup_{j \in J} \mathcal{D}_j$ .

*Proof.* (1) We claim that  $\bigoplus_{\chi \in \mathcal{L}_{<j}} V(\chi) = \bigoplus_{\chi \in \mathcal{L}_{\leq j}} V(\chi)$ . We only have to prove that if  $\chi \in F_j$  then  $V(\chi)$  is a summand of  $\bigoplus_{\chi \in \mathcal{L}_{<j}} V(\chi)$ . Since  $j \notin J$  there is  $\bar{g} \in G/G_e$  such that  $\bar{g}(j) < j$ . Put  $\chi' = \bar{g}(\chi) \in F_{\bar{g}(j)} \subset \mathcal{L}_{<j}$ . Using (3.2), (3.3) we obtain

$$\begin{aligned} V(\chi') &= \mathrm{Ind}_{G_e}^G V_{G_e}(\chi') \\ &= \mathrm{Ind}_{G_e}^G \sigma_{\bar{g}}^{-1} V_{G_e}(\chi) \\ &\cong \mathrm{Ind}_{G_e}^G V_{G_e}(\chi) \\ &= V(\chi). \end{aligned}$$

- (2) The fact that  $\mathcal{D}_{<j} = \mathcal{D}(\Lambda_{<j})$  follows from Lemma 3.5.5. To prove that  $\Lambda_{<j}$  is locally of finite global dimension we may restrict to the case that  $X$  is affine. To prove that  $\mathrm{gl\,dim} \Lambda_{<j} < \infty$ , by [ŠVdB15, Thm 4.3.1, Lem. 4.5.1] it suffices to consider the case  $X = W^\vee$  and  $G = G_e$ . We denote  $\tilde{P}_{\mathcal{L}_{<j},\chi} = \mathrm{Hom}_{X/G}(P_{<j}, P_\chi)$ . By [ŠVdB15, Lem. 11.1.1] it is enough to show that  $\mathrm{pdim} \tilde{P}_{\mathcal{L}_{<j},\chi} < \infty$  for every  $\chi \in X(T)^+$ . Assume that there exists  $\chi$

such that  $\mathrm{pdim} \tilde{P}_{\mathcal{L}_{< j}, \chi} = \infty$  and take  $\chi \in X(T)^+$  with minimal  $(r_\chi, |\mathbf{S}_\chi|)$ . Then  $(r_\chi, \mathbf{S}_\chi) \not\leq (r_\mu, \mathbf{S}_\mu)$  for all  $\mu \in \mathcal{L}_{< j}$  (for otherwise  $\chi \in \mathcal{L}_{< j}$  and hence  $\mathrm{pdim} \tilde{P}_{\mathcal{L}_{< j}, \chi} = 0$ ). Let  $\lambda = \lambda_\chi$ . It follows from Lemma 5.2.5 (1) that  $\langle \lambda, \chi \rangle < \langle \lambda, \mu \rangle$  for all  $\mu \in \mathcal{L}_{< j}$ . Thus,  $C_{\mathcal{L}_{< j}, \lambda, \chi} := \mathrm{Hom}_{X/G}(P_{\mathcal{L}_{< j}}, C_{\lambda, \chi})$  is acyclic by (4.12) and the fact that the  $\lambda$ -weights of  $k[X^{\lambda, +}]$  are  $\leq 0$  (see §3.2). We have  $(r_{\chi'}, |\mathbf{S}_{\chi'}|) < (r_\chi, |\mathbf{S}_\chi|)$  for all  $\tilde{P}_{\mathcal{L}_{< j}, \chi'} \neq \tilde{P}_{\mathcal{L}_{< j}, \chi}$  that appear in  $C_{\mathcal{L}_{< j}, \lambda, \chi}$  by (4.14) and Corollary 5.2.6. Hence  $\mathrm{pdim} \tilde{P}_{\mathcal{L}_{< j}, \chi'} < \infty$  by the minimality assumption, and therefore  $\mathrm{pdim} \tilde{P}_{\mathcal{L}_{< j}, \chi} < \infty$ , a contradiction.

- (3) Now we use the fact that  $(G, B, T, X, \mathcal{L}_j, \chi, \lambda_j)$  is a reduction setting for  $\chi \in F_j$  by Corollary 5.4.5. Let us abbreviate  $D_{j, \chi} = \mathrm{RInd}_{G^{\lambda_j, +}}^G(V_{G^{\lambda_j}}(\chi) \otimes \mathcal{O}_{X^{\lambda_j, +}})$ . By Lemma 4.3.1 we have

$$\pi_{s*} \mathrm{R}\mathcal{E}nd_{X/G}(\oplus_{\chi \in F_j} D_{j, \chi}) = \pi_{s*} \mathcal{E}nd_{X^\lambda/G^\lambda}(\oplus_{\chi \in F_j} P_{G^\lambda, \chi})$$

and the latter is equal to  $\Lambda_{r_j, \lambda_j, \nu_j}$  by Lemma 5.2.7.

To prove that  $\Lambda_{r_j, \lambda_j, \nu_j}$  locally has finite global dimension we may reduce to the affine case. Then by [ŠVdB15, Thm 4.3.1, Lem. 4.5.1] we may reduce to the case  $G = G_e$  and  $X = W^\vee$ . Finally we invoke Proposition 1.2.1.

- (4) The fact that  $\mathcal{D}_{\leq j}$  is generated by  $\mathcal{D}_{< j}$  and  $\mathcal{D}_j$  follows from (4.3). The fact that  $\mathrm{Hom}_{X/G}(\mathcal{D}_{< j}, \mathcal{D}_j) = 0$  follows from (4.1) by a suitable version of the local global spectral sequence on  $X//G$ .
- (5) This follows from Lemma 3.5.4.

*Proof of Theorem 1.1.2.* Put

$$j_0 = \min\{j \in J \mid r_j \geq 1\}.$$

Then we have by Lemma 3.5.5  $\mathcal{D}_{\leq j_0} = \mathcal{D}(\Lambda_{1,0,0})$ . Now write

$$\{j \in J \mid r_j \geq 1\} = \{j_0, j_1, j_2, \dots\}.$$

Put  $\mathcal{D}_0 = \mathcal{D}_{\leq j_0}$  and for  $i > 0$  let  $\mathcal{D}_{-i}$  be the right orthogonal of  $\mathcal{D}_{\leq j_{i-1}} = \mathcal{D}_{< j_i}$  in  $\mathcal{D}_{\mathcal{L}_{\leq j_i}}$ . By Proposition 6.1 (1,4,5) we have a semi-orthogonal decomposition

$$(6.1) \quad \mathcal{D}(X/G) = \langle \dots, \mathcal{D}_{-2}, \mathcal{D}_{-1}, \mathcal{D}_0 \rangle$$

and by Proposition 6.1(3) each of the  $\mathcal{D}_{-1}, \mathcal{D}_{-2}, \dots$  has the required form. The corresponding statement for  $\tilde{\mathcal{D}}_{X/G}$  follows by replacing  $X//G$  by open subschemes.  $\square$

*Remark 6.2.* It follows from Lemma 3.5.7 that the  $\mathcal{D}_{-j} \subset \mathcal{D}(X/G)$  as  $\mathcal{D}_{\leq j}$  are admissible.

*Proof of Proposition 1.2.2.* This corresponds to the special case  $X = W^\vee$  in the proof of Theorem 1.1.2.  $\square$

## 7. THE CASE THAT $X$ DOES NOT HAVE A $T$ -STABLE POINT.

In this section we will assume throughout that  $G$  is connected and that  $X$  is a *connected* smooth  $G$ -variety such that a good quotient  $X//G$  exists. We will give an alternative semi-orthogonal decomposition of  $\mathcal{D}(X/G)$  in case  $X$  does not have a  $T$ -stable point. The results in this section are mostly independent from the rest of the paper.

**Lemma 7.1.** *Assume that  $X$  does not have a  $T$ -stable point. Then at least one of the following settings holds:*

- (1) *There is a non-trivial normal connected subgroup  $K$  of  $G$  acting trivially on  $X$ .*
- (2) *There is a non-trivial central one parameter group  $\nu : G_m \rightarrow Z(G)$  such that  $X = X^{\nu,+}$ .*

*Proof.* Since  $X$  does not have a  $T$ -stable point we have

$$(7.1) \quad X = \bigcup_{\sigma \in X(T) - \{0\}} X^{\sigma,+}.$$

It is well-known and easy to see that there are only a finite number of distinct  $X^{\sigma,+}$ . Indeed: by covering  $X//G$  by a finite number of affines, it suffices to verify this in the affine case and then it follows by embedding  $X$  into a representation. Hence since the union in (7.1) is finite and  $X$  is irreducible we find that there is some  $\sigma \in X(T) - \{0\}$  such that  $X = X^{\sigma,+}$ .

It follows that  $X = X^{w\sigma,+}$  for every  $w \in \mathcal{W}$ . Put  $\nu = \sum_{w \in \mathcal{W}} w\sigma$ . Then  $\nu$  is a  $\mathcal{W}$ -invariant 1-parameter subgroup of  $T$  and in particular its image is contained in the center of  $G$ .

We claim  $X^{\nu,+} = X$ ,  $X^\nu \subset X^\sigma$ . We may check this in the case that  $X$  is affine. Let  $C \subset X(T)_\mathbb{R}$  be the cone spanned by the weights of  $k[X]$ . Since  $X^{\sigma,+} = X$  we have  $\langle \sigma, C \rangle \leq 0$ . Since  $C$  is  $\mathcal{W}$ -invariant we immediately deduce  $\langle \nu, C \rangle \leq 0$  and hence  $X^{\nu,+} = X$ .

To prove  $X^\nu \subset X^\sigma$  we have to verify that if  $\chi \in C$  and  $\langle \nu, \chi \rangle = 0$  then  $\langle \sigma, \chi \rangle = 0$ . To prove this it suffices to observe that  $\langle \nu, \chi \rangle = \sum_{w \in \mathcal{W}} \langle \sigma, w^{-1}\chi \rangle$  and this can only be zero if  $\langle \sigma, w^{-1}\chi \rangle = 0$  for all  $w \in \mathcal{W}$ .

If  $\nu \neq 0$  then we are in situation (2). If  $\nu = 0$  then  $X^\nu = X$ . Hence also  $X^\sigma = X$  by the above discussion. Therefore (1) holds with  $K$  being the identity component of  $\ker(G \rightarrow \text{Aut}(X))$ . The group  $K$  is not trivial as it contains  $\text{im } \sigma$ .  $\square$

To continue it will be convenient to slightly generalize our setting in a similar way as [ŠVdB15, Thm 1.6.3]. We will however use different notations which are more adapted to the current setting. We will assume that  $G$  contains a finite central subgroup  $A$  acting trivially on  $X$  with  $\bar{G} := G/A$ . Let  $X(A) := \text{Hom}(A, G_m)$  be the character group of  $A$ . For  $\tau \in X(A)$  let  $\mathcal{D}(X/G)_\tau$  be the triangulated subcategory of  $\mathcal{D}(X/G)$  consisting of complexes on which  $A$  acts as  $\tau$ . We have an orthogonal decomposition

$$(7.2) \quad \mathcal{D}(X/G) = \bigoplus_{\tau \in X(A)} \mathcal{D}(X/G)_\tau$$

and moreover  $\mathcal{D}(X/G)_0 = \mathcal{D}(X/\bar{G})$ . In general we should think of  $\mathcal{D}(X/G)_\tau$  as a twisted version of  $\mathcal{D}(X/\bar{G})$ .

**Proposition 7.2.** *Let  $\tau \in X(A)$ . Then there exists a semi-orthogonal decomposition of  $\mathcal{D}(X/G)_\tau$  of the form described in Theorem 1.1.2.*

*Proof.* Let the notation be as in §6. We have  $A \subset T$ . Let  $\bar{T} = T/A$ . Then there is an exact sequence

$$0 \rightarrow X(\bar{T}) \rightarrow X(T) \rightarrow X(A) \rightarrow 0$$

Let  $X(\bar{T})_\tau \subset X(T)$  be the inverse image of  $\tau \in X(A)$ . Let  $\chi \in F_j \cap X(\bar{T})_\tau$ . Since  $A$  acts trivially on  $X$ , it acts with the character  $\tau$  on the right-hand side of (4.3) with



$\mathcal{L} = \mathcal{L}_{< j}$ . Since  $\pi_{s*} \mathcal{H}om_{X/G}(P_{G, \chi'}, P_{G, \chi}) = 0$  if  $\chi' \notin X(\bar{T})_\tau$ , (4.3) is thus still an isomorphism if we replace  $\mathcal{L}_{< j}$  by  $\mathcal{L}_{< j, \tau} = \mathcal{L}_{< j} \cap X(\bar{T})_\tau$ . Put

$$\begin{aligned}\Lambda_{< j, \tau} &= \pi_{s*} \mathcal{E}nd_{X/G}(P_{\mathcal{L}_{< j, \tau}}), \\ \mathcal{L}_{j, \tau} &= X(\bar{T})_\tau^{\lambda_j} \cap (\nu - \bar{\rho}_{\lambda_j} + r_j \Sigma_{\lambda_j}^0), \\ U_{j, \tau} &= \bigoplus_{\mu \in \mathcal{L}_{j, \tau}} \text{Ind}_{G_e^{\lambda_j}}^{\bar{G}^{\lambda_j}} V_{G_e^{\lambda_j}}(\mu), \\ \Lambda_{j, \tau} &= (\text{End}(U_{j, \tau}) \otimes \mathcal{O}_{X^{\lambda_j}})^{\bar{G}^{\lambda_j}}\end{aligned}$$

(here  $\bar{G}^\lambda \subset G^\lambda$  as in §3.7, it is not related to  $\bar{G}$ ). We have  $\Lambda_{< j} = \bigoplus_{\tau \in X(A)} \Lambda_{< j, \tau}$ ,  $\Lambda_j = \bigoplus_{\tau \in X(A)} \Lambda_{j, \tau}$ . As  $\Lambda_{< j}$  and  $\Lambda_j$  have finite global dimension when restricted to affine opens in  $X//G$ , the same holds for  $\Lambda_{< j, \tau}$ ,  $\Lambda_{j, \tau}$ . Arguing as above we thus obtain a semi-orthogonal decomposition  $\mathcal{D}_{\leq j, \tau} = \langle \mathcal{D}_{j, \tau}, \mathcal{D}_{< j, \tau} \rangle$  (with the obvious definitions for  $\mathcal{D}_{j, \tau}, \mathcal{D}_{< j, \tau}$ ),  $\mathcal{D}_{< j, \tau} \cong \mathcal{D}(\Lambda_{< j, \tau})$ ,  $\mathcal{D}_{j, \tau} \cong \mathcal{D}(\Lambda_{j, \tau})$ . Then the proof continues as before.  $\square$

If  $K$  is a connected normal subgroup of  $G$  then we will define a *pseudo-complement* of  $K$  as a connected normal subgroup  $Q$  of  $G$  such that  $K$  and  $Q$  commute,  $G = KQ$  and  $K \cap Q$  is finite. It follows easily from [Spr98, Theorem 8.1.5, Corollary 8.1.6] that such a pseudo-complement always exists.

**Proposition 7.3.** *Assume  $X$  does not have a  $T$ -stable point and that we are in the situation of Proposition 7.1(1). Let  $Q$  be a pseudo-complement of  $K$  in  $G$ . Then there is a finite central subgroup  $A_Q$  of  $Q$  acting trivially on  $X$  such that there is an orthogonal decomposition*

$$(7.3) \quad \mathcal{D}(X/G)_\tau \cong \bigoplus_{i \in I} \mathcal{D}(X/Q)_{\mu_i}$$

for a suitable collection of  $(\mu_i)_{i \in I} \in X(A_Q)$ .

*Proof.* Let  $\tilde{G} := K \times Q \rightarrow G$  be the multiplication map and let  $\tilde{A} \subset K \times Q$  be the inverse image of  $A$ . Let  $A_K \subset K$ ,  $A_Q \subset Q$  be the images of  $\tilde{A}$  under the projections  $K \times Q \rightarrow K, Q$ . It is easy to see that  $A_Q$  acts trivially on  $X$ . Let  $\tilde{\tau}$  be the composition  $\tilde{A} \rightarrow A \xrightarrow{\tau} G_m$ . In a similar way, if  $\mu \in X(A_K)$  or  $\mu \in X(A_Q)$  then we denote by  $\tilde{\mu}$  the element of  $X(\tilde{A})$ , obtained by composing  $\mu$  with the appropriate projections  $\tilde{A} \rightarrow A_K, A_Q$ .

We have

$$(7.4) \quad D(X/G)_\tau = D(X/\tilde{G})_{\tilde{\tau}}.$$

Let  $\text{Irr}(K)$  be the set of isomorphism classes of irreducible representations of  $K$ . We have an orthogonal decomposition

$$(7.5) \quad \bigoplus_{V \in \text{Irr}(K)} D(X/Q) \rightarrow D(X/\tilde{G}) : (\mathcal{F}_V)_V \mapsto \bigoplus_{V \in \text{Irr}(K)} V \otimes_k \mathcal{F}_V.$$

If  $V \in \text{Irr}(K)$  then  $A_K$  acts on  $K$  via a character which we denote by  $\chi_V \in X(A_K)$ . Combining (7.4) and (7.5) yields

$$D(X/G)_\tau \cong \bigoplus_{(\mu, V) \in X(A_Q) \times \text{Irr}(K), \tilde{\mu} + \tilde{\chi}_V = \tilde{\tau}} D(X/Q)_\mu$$

which implies (7.3).  $\square$

**Proposition 7.4.** *Assume  $X$  does not have a  $T$ -stable point and that we are in the situation of Proposition 7.1(2). Let  $Q$  be a pseudo-complement of  $\mathrm{im} \nu$  in  $G$ . Then there is a finite central subgroup  $A_Q$  of  $Q$  acting trivially on  $X^\nu$  such that there is a semi-orthogonal decomposition*

$$(7.6) \quad \mathcal{D}(X/G)_\tau = \langle \mathcal{D}(X^\nu/Q)_{\mu_i} \mid i \in I \rangle$$

for a suitable totally ordered set  $I$  and a collection of  $(\mu_i)_{i \in I} \in X(A_Q)$ .

*Proof.* Without loss of generality we may, and we will, assume that  $\nu$  is injective. We put  $K = \mathrm{im} \nu \cong G_m$  and we borrow the associated notation from the proof of Proposition 7.3. One checks that in this case  $A_Q$  acts indeed trivially on  $X^\nu$ . The set  $\mathrm{Irr}(K)$  is equal to  $\{(\chi_n)_n\}$  where  $\chi_n \in X(K)$  is such that  $\chi_n(z) = z^n$ .

We know by Lemma 3.5.4 that  $\mathcal{D}(X/\tilde{G})$  is locally generated by  $P_{n,V} := (\chi_n \otimes V \otimes \mathcal{O}_X)_{n,V}$ , with  $n \in \mathbb{Z}$ ,  $V \in \mathrm{Irr}(Q)$ . We also put  $P_V^\nu := V \otimes \mathcal{O}_{X^\nu} \in \mathcal{D}(X^\nu/Q)$ . We claim that for  $n \leq m$  one has

$$(7.7) \quad \pi_{s,*} \mathrm{RHom}_{X/\tilde{G}}(P_{n,V}, P_{m,V'}) = \begin{cases} 0 & \text{if } n < m \\ \pi_{s,*} j_{s,*} \mathrm{RHom}_{X^\nu/Q}(P_V^\nu, P_{V'}^\nu) & \text{if } n = m \end{cases}$$

where  $j : X^\nu \rightarrow X$  is the embedding and  $j_s : X^\nu/Q \rightarrow X/\tilde{G}$  is the corresponding map of quotient stacks.

To prove this we may reduce to the case that  $X/\tilde{G}$  is affine. Then as usual  $\nu$  induces a grading on  $k[X]$  which, as  $\nu$  is central, is compatible with the  $G$ -action.

In the affine case  $\pi_{s,*} \mathrm{RHom}_{X/\tilde{G}}(P_{n,V}, P_{m,V'})$  is the quasi-coherent sheaf on  $X/\tilde{G}$  associated to the  $k[X]^G$ -module given by  $(\mathrm{Hom}(V, V') \otimes k[X]_{m-n})^{\tilde{G}}$  which is zero if  $n < m$  since the hypothesis  $X = X^{\nu,+}$  implies that the grading on  $k[X]$  is concentrated in negative degree. Similarly if  $n = m$  then we have  $(\mathrm{Hom}(V, V') \otimes k[X]_{m-n})^{\tilde{G}} = (\mathrm{Hom}(V, V') \otimes k[X^\nu])^Q$  finishing the proof of (7.7).

Let  $\mathcal{D}_n \subset \mathcal{D}(X/\tilde{G})$  be locally generated by  $(P_{-n,V})_{V \in \mathrm{Irr}(Q)}$ . Then using Proposition 3.5.8 and (7.7) we see that we have a semi-orthogonal decomposition

$$(7.8) \quad \mathcal{D}(X/\tilde{G}) = \langle \mathcal{D}_n \mid n \in \mathbb{Z} \rangle.$$

The next step is to describe the  $\mathcal{D}_n$ . We claim that there is an equivalence of categories

$$(7.9) \quad \mathcal{D}_n \rightarrow D(X^\nu/Q) : F \mapsto \chi_n \otimes Lj_s^*(F).$$

Let  $F, F' \in \mathcal{D}_n$ . We have to prove that the natural map

$$\mathrm{RHom}_{X/\tilde{G}}(F, F') \rightarrow \mathrm{RHom}_{X^\nu/Q}(\chi_n \otimes Lj_s^*(F), \chi_n \otimes Lj_s^*(F'))$$

is an isomorphism. Using the local global spectral sequence it suffices to prove that

$$\pi_{s,*} \mathrm{RHom}_{X/\tilde{G}}(F, F') \rightarrow \pi_{s,*} Rj_{s,*} \mathrm{RHom}_{X^\nu/Q}(\chi_n \otimes Lj_s^*(F), \chi_n \otimes Lj_s^*(F'))$$

is an isomorphism. To do this we may assume that  $X/\tilde{G}$  is affine. Then we can check it on the generators  $P_{-n,V}$  of  $\mathcal{D}_n$  and finally we invoke 7.7.

Combining the equivalence (7.9) with the semiorthogonal decomposition (7.8) we obtain a semi-orthogonal decomposition

$$\mathcal{D}(X/\tilde{G}) = \langle \mathcal{D}(X^\nu/Q) \mid n \in \mathbb{Z} \rangle$$

Considering suitable subset of the local generators one obtains in the same way a semi-orthogonal decomposition of  $\mathcal{D}(X/\tilde{G})_{\tilde{\tau}}$ . To be more precise consider the following set

$$\mathcal{S} = \{(n, \mu) \in \mathbb{Z} \times A_Q \mid -\tilde{\chi}_n + \tilde{\mu} = \tilde{\tau}\}$$

Let  $\prec$  be the partial ordering on  $\mathcal{S}$  induced from the projection  $\mathcal{S} \rightarrow \mathbb{Z}$ . I.e.  $(n, \mu) \prec (n', \mu')$  if and only if  $n < n'$ . Let  $<$  be a total ordering on  $\mathcal{S}$  which refines  $\prec$ . Then we have a semi-orthogonal decomposition

$$\mathcal{D}(X/\tilde{G})_{\tilde{\tau}} = \langle \mathcal{D}(X^\nu/Q)_\mu \mid (n, \mu) \in \mathcal{S} \rangle.$$

Combining this with the identification (7.4) yields (7.6).  $\square$

*Remark 7.5.* Even if  $A = 0$  (and hence  $\mathcal{D}(X/G)_\tau = \mathcal{D}(X/G)$ ), the group  $A_Q$  and the twisting characters  $\mu_i$  will generally be non-trivial in Propositions 7.3, 7.4.

*Remark 7.6.* Note that in Propositions 7.3, 7.4 we have  $\dim Q < \dim G$ . Thus we have made genuine progress. By repeatedly applying Propositions 7.3, 7.4 we reduce to a semi-orthogonal decomposition of  $\mathcal{D}(X/G)_\tau$  involving a set of  $\mathcal{D}(X'/G')_{\tau'}$  such that  $X'$  has a  $T'$ -stable point for  $T'$  a maximal torus of  $G'$ , thus justifying Remark 1.2.5 (and also making it more precise).

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*E-mail address*, Špela Špenko: `Spela.Spenko@ed.ac.uk`

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, THE KING'S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, SCOTLAND, UK

*E-mail address*, Michel Van den Bergh: `micHEL.vandenbergh@uhasselt.be`

DEPARTMENT OF MATHEMATICS, UNIVERSITEIT HASSELT, MARTELARENLAAN 42, 3500 HASSELT, BELGIUM